

# Overlapping Credit Portfolios

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## Abstract

We present an accurate analytical approximation for a joint distribution function of loss of two overlapping credit portfolios using the multidimensional saddlepoint method. The same method is applied to the tail probability of loss from two tranches and to CDO-squared. Numerical examples show that the default correlations effectively destroy non-normal tails, making the conditional normal approximation viable in many practical cases.

## Introduction

Consider  $M$  credit portfolios formed from a universe of  $N$  risky assets. Denote  $L_p(T)$  the loss sustained by portfolio  $p$  by the time horizon  $T$ , and assume that each portfolio hosts a tranche with attachment points  $d_p < u_p$ . In this paper we focus on the following measures of the joint distribution of portfolio loss:

- Joint tail probability

$$P(K_1, \dots, K_M) = P[L_1 \geq K_1, \dots, L_M \geq K_M]. \quad (1)$$

- Tail probability of total tranche loss

$$P[S(L_1, \dots, L_M) \geq K], \text{ where } S = \sum_p ((L_p - d_p)^+ - (L_p - u_p)^+). \quad (2)$$

- CDO-squared (CDO<sup>2</sup>) stop-loss

$$E[(S(L_1, \dots, L_M) - K)^+]. \quad (3)$$

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The joint tail probability provides detailed guidance for the simultaneous credit risk present in correlated portfolios. The tail probability of total tranche loss is a risk measure for a portfolio of single tranche CDOs. The CDO<sup>2</sup> stop-loss is an essential analytical determinant for the valuation of CDOs of CDOs, as shown by Baheti et al (2005). After a discussion of the issues of correlation modeling and reduction to independent assets we proceed to calculate these quantities for  $M = 2$  using the multidimensional saddlepoint method and the conditional normal approximation.

## Default correlation modeling and reduction to independent assets

Let portfolio  $p$  be based on  $N_p$  assets. Because of portfolio overlap,  $N$  is generally less (in practice often significantly less) than the sum  $N_1 + N_2 + \dots + N_M$ . Let  $w_{a,p}$  be the weight of asset  $a$  in the loss exposure of portfolio  $p$ . Loss in the portfolio  $p$  sustained by the time horizon  $T$  can be expressed in terms of the asset default indicators  $U_a(T)$ ,

$$L_p(T) = \sum_a w_{a,p} U_a(T). \quad (4)$$

Each default indicator  $U_a(T)$  is a random variable taking values 0 and 1. Its complement,  $1 - U_a(T)$  is the survival indicator. The marginal distribution of each default indicator is fully determined by its expected value

$$E[U_a(T)] = \mu_a(T), \quad (5)$$

which has the meaning of the default probability of the asset  $a$ . A complete description of the joint distribution function of  $N$  default indicators at a common time horizon  $T$  would be provided by a set of probabilities of all  $2^N$  possible survival outcomes at time  $T$ . For every subset  $\mathcal{C}$ , the probability that all the assets in this subset defaulted and all the others survived is given by the expected value of the product of corresponding default and survival indicators

$$P_{\mathcal{C}}(T) = E\left[\prod_{a \in \mathcal{C}} U_a(T) \prod_{a' \notin \mathcal{C}} (1 - U_{a'}(T))\right]. \quad (6)$$

If the default events were independent, this would reduce to a product of default and survival probabilities at the relevant time horizon

$$P_{\mathcal{C}}(\mu_1, \mu_2, \dots, \mu_M) = \prod_{a \in \mathcal{C}} \mu_a \prod_{a' \notin \mathcal{C}} (1 - \mu_{a'}). \quad (7)$$

However, the assumption of default independence contradicts both the economical intuition and the numerical evidence from tradable financial instruments sensitive to default correlations, such as CDOs and basket default swaps.

A generic way to equip the model with default correlations is to take a probabilistic mixture of independent scenarios. Each scenario is characterized by a certain combination of asset survival probabilities  $\{\mu_a(T)\}$  at all time horizons relevant for the problem. Default events remain independent in each scenario, and default correlations appear as a result of averaging

over scenarios and depend on the weights assigned to the different scenarios in the mixture. Burtschell et al (2005) gave an explicit description of the standard Gaussian copula model and other copula models in terms of continuous probability distributions for marginal default probabilities and argued in favor of using those distributions as a starting point for correlation modeling instead of the copula functions. Hull and White (2005) made a radical step suggesting the replacement of continuous mixtures of infinitely many scenarios by discrete mixtures of a finite number of scenarios. The immediate advantage of using discrete probability distributions on a judiciously chosen set of scenarios is the ease of exact calibration to a sparse set of data provided by the tradable instruments.

In the framework of copula functions the probability measures on survival probability scenarios for all assets at different time horizons come in tightly bound artificial packages controlled by a small number of numerical parameters, in the extreme case by just one effective correlation. For example, the standard one-factor Gaussian copula model with the correlation  $\rho$  generates a probability measure  $dP(\mu)$  for the default probability  $\mu$  of every asset in the basket from the assumed normal density  $dP(X) = n(X)dX = (1/\sqrt{2\pi}) \exp(-X^2/2)dX$  of the global factor  $X$  in accordance with the relationship

$$\mu(X) = \mathcal{N}\left(\frac{\mathcal{N}^{-1}(\mu_0) - \sqrt{\rho}X}{\sqrt{1-\rho}}\right), \quad (8)$$

where  $\mu_0$  is the expected default probability of the asset,  $\mathcal{N}(x) = \int_{-\infty}^x n(t)dt$  is the normal cumulative distribution function (CDF), and  $\mathcal{N}^{-1}$  is the inverse normal CDF. The explicit expression for the probability density of default probability reads

$$dP(\mu) = \sqrt{\frac{1-\rho}{\rho}} \exp\left[-\frac{(1-2\rho)(\mathcal{N}^{-1}(\mu))^2 - 2\sqrt{1-\rho}\mathcal{N}^{-1}(\mu_0)\mathcal{N}^{-1}(\mu) + (\mathcal{N}^{-1}(\mu_0))^2}{2\rho}\right] d\mu, \quad (9)$$

plotted in Fig. 1 for several values of correlations. For  $\rho < 50\%$  this density has a maximum at  $\mu = \mathcal{N}(\frac{\sqrt{1-\rho}}{1-2\rho}\mathcal{N}^{-1}(\mu_0))$  and falls off to zero at the bounds  $\mu = 0,1$ . In the limit  $\rho \rightarrow 0$  the distribution of default probability becomes sharply peaked at  $\mu = \mu_0$ . For  $\rho > 50\%$  the distribution is bimodal with infinite density at  $\mu = 0,1$  and a minimum in between. The crossover value  $\rho = 50\%$  corresponds to a density decreasing from infinity at  $\mu = 0$  to zero at  $\mu = 1$ .

The single factor Gaussian copula function also fixes the joint distribution of the default probability of all the assets which have a positive correlation with the global factor by imposing a rigid constraint on the default probabilities. A particular value of the default probability for one of the assets at one time horizon uniquely determines the global factor  $X$  and consequently the default probabilities of all assets at all time horizons. While there is nothing apparently pathological in the probability density (9) (at least for  $\rho < 50\%$ ), Hull and White (2005) argued that the induced joint distribution of default probabilities of the same asset at different time horizons turned out to be unrealistic. Specifically they showed that the hazard rate paths in some of the scenarios exhibited the pattern of credit risk decreasing with time. We would not rule out the presence of a certain fraction of scenarios with decreasing credit risk in the mixture as incompatible with the real world (after all there is a non-zero probability that some or even all credits will improve with time). However, the phenomenon of the correlation skew, discussed at length by Andersen and Sidenius (2005) and Burtschell et al (2005), proves beyond doubt that the single factor Gaussian copula at the very least does not capture this fraction correctly.

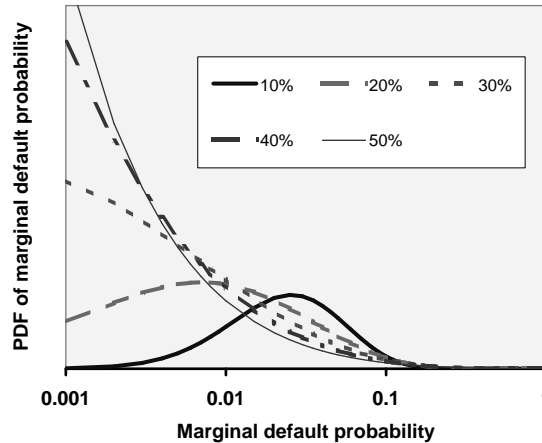


Figure 1: Probability densities (9) of marginal default probability of an asset generated by the one-factor Gaussian copula model with different correlations. The average default probability is 0.05.

The distribution of the upper tail of portfolio loss is strongly affected by the weight assigned to scenarios with probabilities of defaults much larger than the average. In the framework of the discrete perfect copula model the qualitative behavior of the tail probability  $P[L \geq K]$  can be understood from the presence or absence of scenarios with expected loss of the order of  $K$  or larger with a non-negligible weight. If such scenarios are present, the distribution tail is dominated by frequent defaults in these scenarios and has no room to deviate from normal significantly. Otherwise the tail probability has to come entirely from the rare events of simultaneous defaults of many uncorrelated assets of low risk. In this case, the tail of the loss distribution is strongly non-normal and requires more advanced techniques, such as saddlepoint, to estimate. The same two regimes arise in the continuous copula framework. We will see that for the standard Gaussian copula the non-normal regime of vanishing correlations quickly crosses over to the normal regime with increasing correlation.

## Distribution of loss from overlapping portfolios of independent assets

The upshot of the previous section is that regardless of the particular approach to correlation modeling we need to solve the auxiliary problem of loss distribution for independent assets. This is our focus in this section. The calculations assume a scenario with a certain set  $\{\mu_a\}$  of asset default probabilities.

### Summary of analytical results for a single portfolio

We proceed with a summary of results for the probability distribution of loss  $L(T)$  for a single portfolio. The starting point is the decomposition in asset default indicators,

$$L(T) = \sum_a w_a U_a(T). \quad (10)$$

Concise analytical expressions for the quantities of interest are obtained in terms of the cumulant generating function (CGF) defined as  $\mathcal{K}(\xi) = \ln(E[\exp(\xi L)])$  and evaluated explicitly as

$$\mathcal{K}(\xi) = \sum_a \ln(1 - \mu_a + \mu_a e^{\xi w_a}). \quad (11)$$

The tail probability and stop-loss can be restored from the CGF by means of an inverse Laplace transform,

$$P[L \geq K] = \frac{1}{2\pi i} \int_{C^+} \frac{\exp(\mathcal{K}(\xi) - \xi K)}{\xi} d\xi, \quad (12)$$

$$E[(L - K)^+] = \frac{1}{2\pi i} \int_{C^+} \frac{\exp(\mathcal{K}(\xi) - \xi K)}{\xi^2} d\xi. \quad (13)$$

Here  $C^+$  is any infinite contour of integration in the complex plane going along the imaginary axis and crossing the positive half of the real axis. A possible choice of the contour is  $c + it$  with any constant real  $c > 0$  and real  $t$  varying from  $-\infty$  to  $+\infty$ .

The saddlepoint approximation consists in expanding the CGF around the solution of  $\xi = s$  of the equation  $\mathcal{K}'(\xi) = K$  and truncating the expansion at the quadratic term. Its application for the problems of credit portfolios was suggested by Arvanitis and Gregory (1999) and extensively explored by Martin et al (2001). The result is

$$P[L \geq K] = \theta(-s) + \text{sign}(s) e^{\mathcal{K}(s) - sK + \frac{1}{2} m s^2} \mathcal{N}(-\sqrt{m}|s|), \quad (14)$$

$$E[(L - K)^+] = \theta(-s) \left( \sum_a w_a \mu_a - K \right) + e^{\mathcal{K}(s) - sK} g(m, s),$$

$$g(m, s) = \sqrt{m/2\pi} - m|s| e^{\frac{1}{2} m s^2} \mathcal{N}(-\sqrt{m}|s|), \quad m = \mathcal{K}''(s), \quad (15)$$

where  $\text{sign}(x)$  is the sign function equal to 1 for  $x \geq 0$  and -1 for  $x < 0$ , and  $\theta(x)$  is the step function equal to 1 for  $x \geq 0$  and 0 for  $x < 0$ . Antonov et al (2005) computed the correction to the saddlepoint approximation coming from the cubic term  $\mathcal{K}'''(\xi)$  and showed that it can improve the accuracy, however in this paper we will not go beyond second derivatives of CGF.

Alternatively, the CGF can be expanded around  $\xi = 0$ , which is a standard way of deriving central limit theorems. The expansion to the quadratic terms yields the normal approximation for the distribution of portfolio loss with the expected value  $\Lambda$  and variance  $M_2$  given by the expressions

$$\Lambda = \sum_a w_a \mu_a, \quad (16)$$

$$\sigma = \sum_a w_a^2 \mu_a (1 - \mu_a). \quad (17)$$

The values of tail probability and stop-loss resulting from this approximation are

$$P[L \geq K] = \mathcal{N}\left(\frac{\Lambda - K}{\sqrt{\sigma}}\right), \quad (18)$$

$$E[(L - K)^+] = (\Lambda - K) \mathcal{N}\left(\frac{\Lambda - K}{\sqrt{\sigma}}\right) + \sqrt{\frac{\sigma}{2\pi}} \exp\left(-\frac{(\Lambda - K)^2}{2\sigma}\right). \quad (19)$$

Shelton (2004) considered the normal approximation in the framework of copulas where it is conditional on the global factor. More generally the approximation is conditional on the values of asset default probabilities.

For  $\Lambda = K$  the saddlepoint solution is  $s = 0$ , thus we should expect a fairly minor difference between the saddlepoint approximation and the normal approximation for scenarios with expected loss  $\Lambda$  of the order of  $K$ . On the other hand, the standard wisdom is that the normal approximation should work badly in the deep tail of the distribution,  $\Lambda \ll K$ . This is indeed so except for larger values of default probabilities in which case the deep tail does not exist, and the normal approximation remains as accurate as the saddlepoint for all meaningful values of the cutoff  $K$ . We emphasize that the resurgence of the normal distribution and normal CDF in the results of this section is generic and has nothing to do with Gaussian copula assumptions.

## Tail probability for the joint distribution of several portfolios

The starting point for the evaluation of the joint tail probability is a multidimensional generalization of Eq. (12),

$$P(K_1, \dots, K_M) = \frac{1}{(2\pi i)^M} \int_{C_1^+} \frac{d\xi_1}{\xi_1} \dots \int_{C_M^+} \frac{d\xi_M}{\xi_M} \exp(\mathcal{K}(\xi_1, \dots, \xi_M) - \sum_p \xi_p K_p), \quad (20)$$

with the multidimensional CGF  $\mathcal{K}(\xi_1, \dots, \xi_M)$

$$\mathcal{K}(\xi_1, \dots, \xi_M) = \ln E\left[\sum_p \xi_p L_p\right] = \sum_a \ln(1 - \mu_a + \mu_a e^{\sum_p \xi_p w_{a,p}}). \quad (21)$$

The saddlepoint is a point  $(s_1, \dots, s_M)$  where the gradient of the exponent vanishes,

$$\frac{\partial \mathcal{K}}{\partial \xi_p}(s_1, \dots, s_M) = K_p, \quad p = 1, \dots, M. \quad (22)$$

The leading behavior of the exponent near the saddlepoint is determined by the quadratic form of second derivatives

$$m_{pq} = \frac{\partial^2 \mathcal{K}}{\partial \xi_p \partial \xi_q}(s_1, \dots, s_M). \quad (23)$$

Explicit expressions for the first and second derivatives of the CGF and steps of the derivation are relegated to the Appendix. Here we give the final result for the joint tail probability of two portfolios

$$P[L_1 \geq K_1, L_2 \geq K_2] = -\theta(-s_1)\theta(-s_2) + \theta(-s_1)P[L_2 \geq K_2] + \theta(-s_2)P[L_1 \geq K_1] + \text{sign}(s_1 s_2) e^\kappa \Phi_2(h_1, h_2; r), \quad (24)$$

where

$$\kappa = \mathcal{K}(s_1, s_2) - s_1 K_1 - s_2 K_2 + \frac{1}{2} m_{11} s_1^2 + \frac{1}{2} m_{22} s_2^2 + m_{12} s_1 s_2, \quad (25)$$

$$r = \text{sign}(s_1 s_2) \frac{m_{12}}{\sqrt{m_{11} m_{22}}}, \quad (26)$$

$$h_1 = -\text{sign}(s_1) \frac{m_{11}s_1 + m_{12}s_2}{\sqrt{m_{11}}}, \quad (27)$$

$$h_2 = -\text{sign}(s_2) \frac{m_{22}s_2 + m_{12}s_1}{\sqrt{m_{22}}}, \quad (28)$$

and  $\Phi_2(h_1, h_2; r)$  is the standard bivariate normal CDF defined as

$$\Phi_2(h_1, h_2; r) = \frac{1}{2\pi\sqrt{1-r^2}} \int_{-\infty}^{h_1} dx \int_{-\infty}^{h_2} dy \exp\left(-\frac{x^2 - 2rxy + y^2}{2(1-r^2)}\right). \quad (29)$$

The marginal tail probabilities of the portfolios  $P[L_p \geq K_p]$  can be computed using the formulas from the previous subsection.

As with the single portfolio problems, a considerably simpler, direct multivariate normal approximation is available. It can be derived by expanding the CGF to quadratic terms around 0. The average and the covariance matrix are as follows:

$$\Lambda_p = \sum_a w_{a,p} \mu_a, \quad (30)$$

$$\sigma_{pq} = \sum_a w_{a,p} w_{a,q} \mu_a (1 - \mu_a). \quad (31)$$

The tail probability is the multivariate normal CDF. In particular, the joint tail probability of loss from two portfolios is

$$P[L_1 \geq K_1, L_2 \geq K_2] = \Phi_2\left(\frac{\Lambda_1 - K_1}{\sqrt{\sigma_{11}}}, \frac{\Lambda_2 - K_2}{\sqrt{\sigma_{22}}}, \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}\right). \quad (32)$$

Numerical experiments show that the saddlepoint method offers a significant improvement over the normal approximation in scenarios with low expected loss. In scenarios with high expected loss the normal approximation works just as well and can be used instead of the saddlepoint. The qualitatively different behavior of the joint tail probability for small, medium, and large probabilities is shown in Figs. 2–4 where the default probability is the same for all assets and is equal to 1% (small), 5% (moderate), or 15% (large).

## Tail probability of total loss of two tranches

An exact result for two portfolios valid for  $K > 0$  is derived in the Appendix,

$$\begin{aligned} P[S(L_1, L_2) \geq K] &= \theta(u_1 - d_1 - K)P[L_1 \geq K + d_1] + \theta(u_2 - d_2 - K)P[L_2 \geq K + d_2] \\ &\quad + \theta(u_2 - d_2 - K) (P[L_1 \geq d_1, L_1 + L_2 \geq K + d_1 + d_2] - P[L_1 \geq d_1, L_2 \geq K + d_2]) \\ &\quad - \theta(u_1 - d_1 - K)P[L_1 \geq K + d_1, L_1 + L_2 \geq K + d_1 + d_2] \\ &\quad + (\theta(u_1 - d_1 - K) - \theta(u_1 - d_1 + u_2 - d_2 - K)) \\ &\quad \quad \times (P[L_1 \geq u_1, L_1 + L_2 \geq K + d_1 + d_2] - P[L_1 \geq u_1, L_2 \geq K - u_1 + d_1 + d_2]) \\ &\quad - (\theta(u_2 - d_2 - K) - \theta(u_1 - d_1 + u_2 - d_2 - K)) \\ &\quad \quad \times P[L_1 \geq K - u_2 + d_1 + d_2, L_1 + L_2 \geq K + d_1 + d_2]. \end{aligned} \quad (33)$$

Note that  $L_1 + L_2$  is the loss of the union of the portfolios 1 and 2 which can be regarded just as another portfolio overlapping (very significantly) with portfolio 1. Thus the ability to compute the joint tail probability for an arbitrary pair of portfolios is sufficient to recover the tail probability of total tranche loss. Numerical comparison of the results is shown in Figs. 5 and 6, which show that the normal approximation becomes completely adequate to the task already at moderate values of default probabilities.

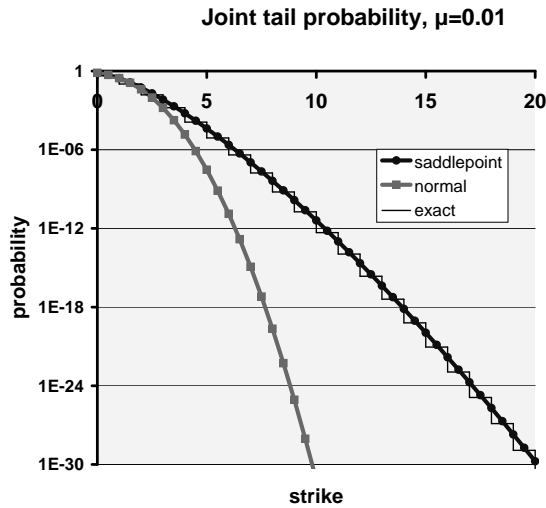


Figure 2: Joint tail probability  $P[L_1 \geq K, L_2 \geq K]$  as a function of strike  $K$  for two portfolios of 100 assets with 24 common assets and unit exposure to each asset,  $w_a = 1$ . All assets are independent and have a small default probability  $\mu = 0.01$ .

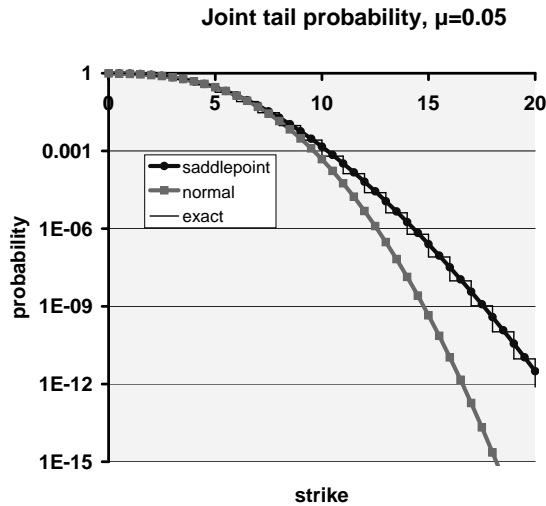


Figure 3: Joint tail probability  $P[L_1 \geq K, L_2 \geq K]$  as a function of strike  $K$  for two portfolios of 100 assets with 24 common assets and unit exposure to each asset,  $w_a = 1$ . All assets are independent and have a moderate default probability  $\mu = 0.05$ .

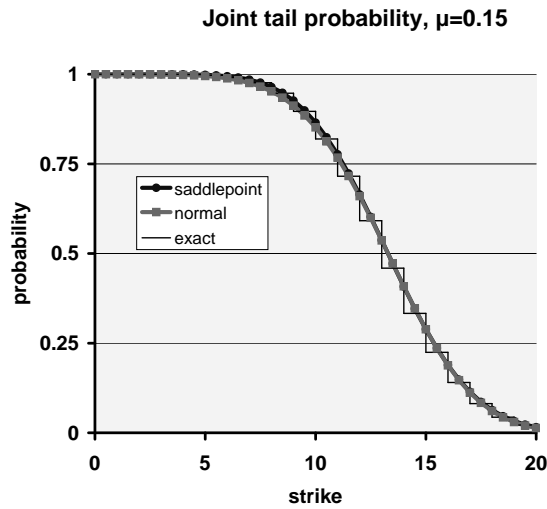


Figure 4: Joint tail probability  $P[L_1 \geq K, L_2 \geq K]$  as a function of strike  $K$  for two portfolios of 100 assets with 24 common assets and unit exposure to each asset,  $w_a = 1$ . All assets are independent and have a large default probability  $\mu = 0.15$ .

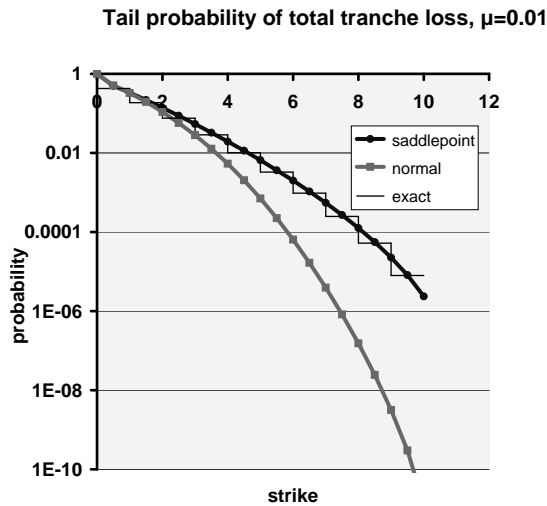


Figure 5: Tail probability  $P[S(L_1, L_2) \geq K]$  where  $S(L_1, L_2) = (L_1 - d_1)^+ - (L_1 - u_1)^+ + (L_2 - d_2)^+ - (L_2 - u_2)^+$  as a function of strike  $K$  for two portfolios of 100 assets with 24 common assets and unit exposure to each asset,  $w_a = 1$ . Low and high bounds are  $d_1 = d_2 = 1$ ,  $u_1 = u_2 = 6$ . All assets are independent and have a small default probability  $\mu = 0.01$ .

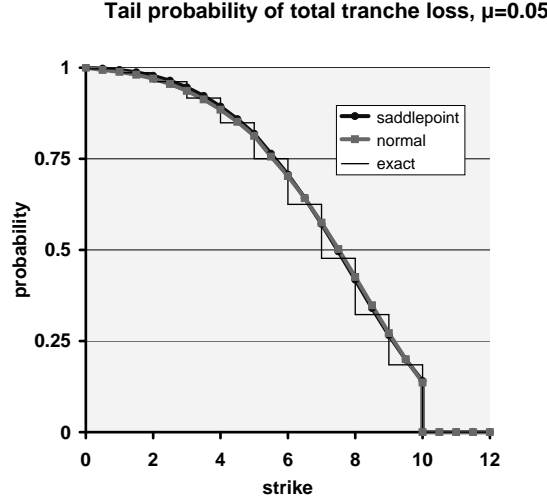


Figure 6: Tail probability  $P[S(L_1, L_2) \geq K]$  where  $S(L_1, L_2) = (L_1 - d_1)^+ - (L_1 - u_1)^+ + (L_2 - d_2)^+ - (L_2 - u_2)^+$  as a function of strike  $K$  for two portfolios of 100 assets with 24 common assets and unit exposure to each asset,  $w_a = 1$ . Low and high bounds are  $d_1 = d_2 = 1$ ,  $u_1 = u_2 = 6$ . All assets are independent and have a moderate default probability  $\mu = 0.05$ .

## CDO<sup>2</sup> stop-loss for two tranches

Just as the tail probability of total tranche loss from two portfolios is expressed as a linear combination of the tail probabilities  $P[L_1 \geq A_1, L_2 \geq A_2]$  taken with different values of cutoffs  $A_1, A_2$  (with the union of the portfolios 1 and 2 in place of the second portfolio in some of the terms), the stop-loss can be expressed in terms of an auxiliary average with the meaning of a mixture of tail probability in one variable and stop-loss in another variable,

$$E[\theta(L_1 - A_1)(L_2 - A_2)^+]. \quad (34)$$

The exact decomposition of the stop-loss is as follows

$$\begin{aligned} E[(S(L_1, L_2) - K)^+] = & \\ & \theta(u_1 - d_1 - K)(E[(L_1 - K - d_1)^+] - E[(L_1 - u_1)^+]) \\ & + \theta(u_2 - d_2 - K)(E[(L_2 - K - d_2)^+] - E[(L_2 - u_2)^+]) \\ & + \theta(u_2 - d_2 - K)(E[\theta(L_1 - d_1)(L_1 + L_2 - K - d_1 - d_2)^+] \\ & \quad - E[\theta(L_1 - d_1)(L_2 - K - d_2)^+]) \\ & - \theta(u_1 - d_1 - K)(E[\theta(L_1 - K - d_1)(L_1 + L_2 - K - d_1 - d_2)^+] \\ & \quad - E[\theta(L_1 - K - d_1)(L_2 - d_2)^+]) \\ & + (\theta(u_1 - d_1 - K) - \theta(u_1 - d_1 + u_2 - d_2 - K)) \\ & \quad \times (E[\theta(L_1 - u_1)(L_1 + L_2 - K - d_1 - d_2)^+] \\ & \quad - E[\theta(L_1 - u_1)(L_2 - K + u_1 - d_1 - d_2)^+]) \\ & - (\theta(u_2 - d_2 - K) - \theta(u_1 - d_1 + u_2 - d_2 - K)) \\ & \quad \times (E[\theta(L_1 - K + u_2 - d_1 - d_2)(L_1 + L_2 - K - d_1 - d_2)^+] \\ & \quad - E[\theta(L_1 - K + u_2 - d_1 - d_2)(L_2 - u_2)^+]). \end{aligned} \quad (35)$$

In order to apply this decomposition numerically we need a method to compute the auxiliary average (34). As with the tail probability, this can be done using the saddlepoint or the

multivariate normal approximation. The saddlepoint evaluation ultimately leads to an average over a new multivariate normal distribution, adapted to the saddlepoint location. We are led to introduce a new standard function  $\Psi_2(h_1, h_2; r)$  similar to the bivariate normal CDF (29) but adapted to the stop-loss problem,

$$\Psi_2(h_1, h_2; r) = \frac{1}{2\pi\sqrt{1-r^2}} \int_{-\infty}^{h_1} dx \int_{-\infty}^{h_2} dy (h_2 - y) \exp\left(-\frac{x^2 - 2rxy + y^2}{2(1-r^2)}\right). \quad (36)$$

Even though this function is not found in standard numerical packages it is no more difficult to implement than the bivariate normal CDF, for example, as a numerical quadrature with the Gaussian density

$$\Psi_2(h_1, h_2; r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{h_2} dy (h_2 - y) e^{-y^2/2} \mathcal{N}\left(\frac{h_1 - ry}{\sqrt{1-r^2}}\right). \quad (37)$$

The requisite average in the bivariate normal approximation is given by

$$E[\theta(L_1 - K_1)(L_2 - K_2)^+] = \sqrt{\sigma_{22}} \Psi_2\left(\frac{\Lambda_1 - K_1}{\sqrt{\sigma_{11}}}, \frac{\Lambda_2 - K_2}{\sqrt{\sigma_{22}}}, \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}\right). \quad (38)$$

To give the corresponding expression in the saddlepoint approximation we need an auxiliary result for a simpler average  $E[\theta(L_1 - K_1)(L_2 - K_2)]$  (in which the difference  $L_2 - K_2$  is not bounded by zero from below). Applying the saddlepoint approximation to the integral representation

$$E[\theta(L_1 - K_1)(L_2 - K_2)] = \frac{1}{2\pi i} \int_{C^+} d\xi_1 \left(\frac{\partial \mathcal{K}}{\partial \xi_2}(\xi_1, 0) - K_2\right) \frac{\exp \mathcal{K}(\xi_1, 0) - \xi_1 K_1}{\xi_1}, \quad (39)$$

we introduce a solution  $\xi_1 = s$  for a scalar saddlepoint equation  $\partial \mathcal{K}(\xi_1, 0)/\partial \xi_1 = K_1$ , move the contour of integration to pass through  $s$ , and expand around the saddlepoint. The result is

$$E[\theta(L_1 - K_1)(L_2 - K_2)] = \theta(-s)(\Lambda_2 - K_2) + e^{\mathcal{K}(s,0) - sK_1} \left( (f_0 - sf_1) \text{sign}(s) e^{\frac{1}{2}ms^2} \mathcal{N}(-\sqrt{m}|s|) + f_1/\sqrt{2\pi m} \right), \quad (40)$$

where

$$m = \frac{\partial^2 \mathcal{K}}{\partial \xi_1^2}(s, 0), \quad (41)$$

$$f_0 = \frac{\partial \mathcal{K}}{\partial \xi_2}(s, 0) - K_2, \quad (42)$$

$$f_1 = \frac{\partial^2 \mathcal{K}}{\partial \xi_1 \partial \xi_2}(s, 0). \quad (43)$$

Finally the saddlepoint result for the mixture of tail probability and stop loss reads

$$E[\theta(L_1 - K_1)(L_2 - K_2)^+] = -\theta(-s_1)\theta(-s_2)(\Lambda_2 - K_2) + \theta(-s_1)E[(L_2 - K_2)^+] + \theta(-s_2)E[\theta(L_1 - K_1)(L_2 - K_2)] + \text{sign}(s_1)\sqrt{m_{22}}e^\kappa \Psi_2(h_1, h_2; r), \quad (44)$$

with  $m_{22}$ ,  $\kappa$ ,  $r$ ,  $h_1$ ,  $h_2$  defined by Eqs. (23) and (25–28). Numerical results of Figs. 7 and 8 illustrate the well expected by now conclusion that the normal approximation becomes valid for moderate values of default probabilities.

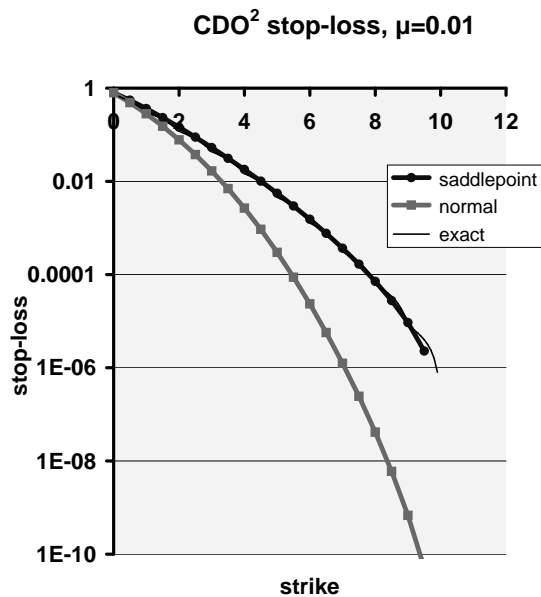


Figure 7: Stop-loss  $E[(S(L_1, L_2) - K)^+]$  where  $S(L_1, L_2) = (L_1 - d_1)^+ - (L_1 - u_1)^+ + (L_2 - d_2)^+ - (L_2 - u_2)^+$  as a function of strike  $K$  for two portfolios of 100 assets with 24 common assets and unit exposure to each asset,  $w_a = 1$ . Low and high bounds are  $d_1 = d_2 = 1$ ,  $u_1 = u_2 = 6$ . All assets are independent and have a small default probability  $\mu = 0.01$ .

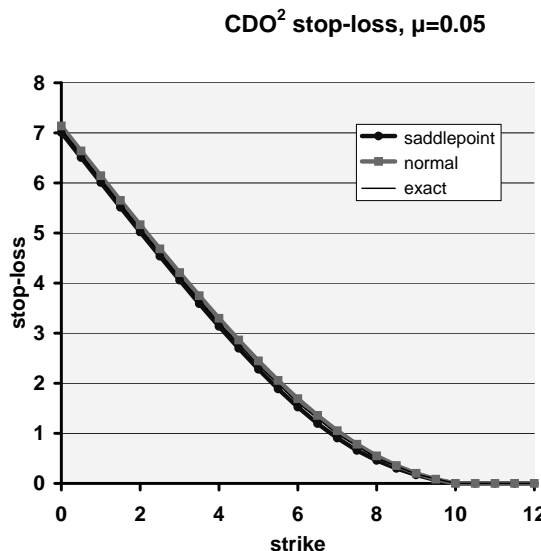


Figure 8: Stop-loss  $E[(S(L_1, L_2) - K)^+]$  where  $S(L_1, L_2) = (L_1 - d_1)^+ - (L_1 - u_1)^+ + (L_2 - d_2)^+ - (L_2 - u_2)^+$  as a function of strike  $K$  for two portfolios of 100 assets with 24 common assets and unit exposure to each asset,  $w_a = 1$ . Low and high bounds are  $d_1 = d_2 = 1$ ,  $u_1 = u_2 = 6$ . All assets are independent and have a moderate default probability  $\mu = 0.05$ .

**Ratio of saddlepoint and normal approximations  
for tail probability, average default probability 1%**

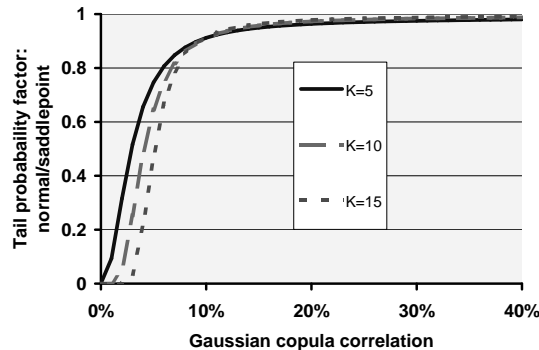


Figure 9: Normal approximation underestimates the joint tail probability  $P[L_1 \geq K, L_2 \geq K]$ . The relative error decreases with increasing correlation and the ratio tends to one. Two portfolios of 100 assets with 24 common assets and unit exposure to each asset,  $w_a = 1$ , and three values of strike  $K = 5, 10, 15$  were used. The model of correlations is the one-factor Gaussian copula.

## Effect of correlations

As argued in the beginning of the paper, the presence of correlations is modeled by a mixture of scenarios with different probabilities of independent defaults. For a standard single factor Gaussian copula model the scenarios with high default probabilities easily win the dominance with the value of the correlation only slightly above zero. Fig.9 illustrates this point for the joint tail probability of two portfolios. The convergence to the normal limit for CDO<sup>2</sup> type problems is only faster because the upper attachment points of the tranches effectively cut the relevant part of the tail.

## Conclusions

We have shown how the analytical saddlepoint techniques can be applied for accurate calculation of measures of joint distribution of loss from several credit portfolios. We also demonstrated that the conditional multivariate normal approximation, while being much simpler both conceptually and computationally, leads to the results of similar accuracy, except for the case of dominance by scenarios with very low default probabilities. The non-normal regime can arise in the case of high quality portfolios and short time horizons. A typical task of CDO and CDO<sup>2</sup> valuation should be well inside the normal regime, which justifies the use of conditional normal approximation of Shelton (2005) to ease the computational burden. Additional techniques for accelerating the calculations in the normal approximation were discussed by Antonov et al (2005).

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## Appendix: Details of saddlepoint calculations

This appendix contains a few intermediate steps of the derivation of the saddlepoint approximation results. The first and second derivatives of the CGF are as follows

$$\frac{\partial \mathcal{K}}{\partial \xi_p} = \sum_a \frac{w_{a,p} \mu_a \exp \sum_r \xi_r w_{a,r}}{1 - \mu_a + \mu_a \exp \sum_r \xi_r w_{a,r}}, \quad (45)$$

$$\frac{\partial^2 \mathcal{K}}{\partial \xi_p \partial \xi_q} = \sum_a \frac{w_{a,p} w_{a,q} \mu_a (1 - \mu_a) \exp \sum_r \xi_r w_{a,r}}{(1 - \mu_a + \mu_a \exp \sum_r \xi_r w_{a,r})^2}. \quad (46)$$

The matrix of the second derivatives is positive definite, and the saddlepoint solution is found easily by an iterative solver. If all the components of the saddlepoint are positive,  $s_p > 0$ ,  $p = 1, \dots, M$ , the contours of integration in Eq. (20) can be moved to pass through the saddlepoint without touching the singularities. The expansion of the exponent around the saddlepoint to the quadratic terms yields

$$\begin{aligned} \mathcal{K}(\xi_1, \dots, \xi_M) - \sum_p \xi_p K_p &= \mathcal{K}(s_1, \dots, s_M) - \sum_p s_p K_p + \sum_{p,q} \frac{1}{2} m_{pq} s_p s_q \\ &\quad + \sum_{p,q} \frac{1}{2} m_{pq} \xi_p \xi_q - \sum_p \xi_p \sum_q m_{pq} s_q + \dots \end{aligned} \quad (47)$$

The first line in the right-hand side of Eq. (47) gives rise to a constant factor while the integral of the exponential of the second line can be immediately reinterpreted as a multivariate normal CDF with covariance matrix  $m_{pq}$ .

An immediate expansion is impossible if one or several components of the saddlepoint are negative. The corresponding contours of integration  $C_p^+$  must first be dragged through the pole at 0 to cross the negative real semi-axis and become  $C_p^-$ . We denote  $\zeta_p = \text{sign}(s_p)$  and introduce the notation  $C_p^{\zeta_p}$  to stand for  $C_p^+$  if  $\zeta_p = 1$  and  $C_p^-$  for  $\zeta_p = -1$ . We work it out explicitly for  $M = 2$ , first deforming the  $C_1^+$  into  $C_1^{\zeta_1}$  and separating the residue term and then doing similarly with  $C_2^+$ . One-dimensional integrals are reinterpreted as tail probabilities using Eq. (12) (for which it may be necessary to deform the contour back picking up a residue

term again).

$$\begin{aligned}
P[L_1 \geq K_1, L_2 \geq K_2] &= \frac{1}{(2\pi i)^2} \int_{C_1^+} \frac{d\xi_1}{\xi_1} \int_{C_2^+} \frac{d\xi_2}{\xi_2} \exp(\mathcal{K}(\xi_1, \xi_2) - \xi_1 K_1 - \xi_2 K_2) \\
&= \frac{1}{2\pi i} \theta(-s_1) \int_{C_2^+} \frac{d\xi_2}{\xi_2} \exp(\mathcal{K}(0, \xi_2) - \xi_2 K_2) \\
&\quad + \frac{1}{(2\pi i)^2} \int_{C_1^{\zeta_1}} \frac{d\xi_1}{\xi_1} \int_{C_2^+} \frac{d\xi_2}{\xi_2} \exp(\mathcal{K}(\xi_1, \xi_2) - \xi_1 K_1 - \xi_2 K_2) \\
&= \theta(-s_1) P[L_2 \geq K_2] + \frac{\theta(-s_2)}{2\pi i} \int_{C_1^{\zeta_1}} \frac{d\xi_1}{\xi_1} \exp(\mathcal{K}(\xi_1, 0) - \xi_1 K_1) \\
&\quad + \frac{1}{(2\pi i)^2} \int_{C_1^{\zeta_1}} \frac{d\xi_1}{\xi_1} \int_{C_2^{\zeta_2}} \frac{d\xi_2}{\xi_2} \exp(\mathcal{K}(\xi_1, \xi_2) - \xi_1 K_1 - \xi_2 K_2) \\
&= \theta(-s_1) P[L_2 \geq K_2] + \theta(-s_2) P[L_1 \geq K_1] - \theta(-s_1) \theta(-s_2) \\
&\quad + \frac{1}{(2\pi i)^2} \int_{C_1^{\zeta_1}} \frac{d\xi_1}{\xi_1} \int_{C_2^{\zeta_2}} \frac{d\xi_2}{\xi_2} \exp(\mathcal{K}(\xi_1, \xi_2) - \xi_1 K_1 - \xi_2 K_2).
\end{aligned} \tag{48}$$

The expansion (47) around the saddlepoint in the resulting double integral has become possible. In order to restore the bivariate normal CDF after the expansion we need to return back to the original contours  $C_1^+$ ,  $C_2^+$ . An easy way to do it is to make a change of variables  $\xi_p \rightarrow \zeta_p \xi_p$ . This completes the derivation of the joint tail probability of two portfolios in the saddlepoint approximation, Eq. (24).

The derivations of the decompositions for the tail probability of total tranche loss and CDO<sup>2</sup> stop-loss can be done by a careful inspection of the regions in the  $M$ -dimensional space of portfolio loss  $(L_1, \dots, L_M)$  cut by the various planes representing the conditions when the contribution of each combination of tranches “kicks in.” This is a very laborious exercise even in the case of two portfolios. There is a fairly automatic alternative way to generate all the terms starting from the integral representations derived by Antonov et al (2005), which employs an auxiliary variable  $y$  and one more integration contour  $C_y^+$  that crosses the positive real semi-axis,

$$\begin{aligned}
P[\theta(S - K)(S - K)^\alpha] &= \int_{C_1^+} \cdots \int_{C_M^+} \int_{C_y^+} \frac{dy \prod d\xi_p}{(2\pi i)^{M+1}} \frac{\exp(-yK)}{y^{1+\alpha}} \exp \mathcal{K}(\xi_1, \dots, \xi_M) \\
&\quad \prod_p \left( \frac{1}{\xi_p} + \frac{y [\exp(y(u_p - d_p) - \xi_p u_p) - \exp(-\xi_p d_p)]}{(y - \xi_p) \xi_p} \right).
\end{aligned} \tag{49}$$

The result does not depend on the particular sequence of the contours  $C_1^+$ ,  $\dots$ ,  $C_M^+$ , and  $C_y^+$ , but the same sequence has to be kept for all the terms that appear after expanding the products. It is convenient to take  $C_y^+$  as the rightmost contour. Then the integral of every term of the form  $f(y)e^{y(A-K)}$  where  $f(y)$  is a ratio of two polynomials in  $y$  that tends to 0 as  $y \rightarrow \infty$  vanishes for  $K \geq A$  (because contour  $C_y^+$  can be closed to the right) and can be easily computed from the residues at the poles  $y = 0, \xi_1, \dots, \xi_M$  for  $K < A$  (because contour  $C_y^+$  can be closed to the left). Eqs. (33) and (35) are obtained in the case of two portfolios after cumbersome but straightforward calculations using a substitution

$$\exp \mathcal{K}(\xi_1, \xi_2) = E[\exp(\xi_1 L_1 + \xi_2 L_2)], \tag{50}$$

with the subsequent integration first over  $\xi_1$  and then over  $\xi_2$ . Note that the entire sequence of transformations leading to Eqs. (33) and (35) is exact. Approximations appear only in the evaluation of joint tail probability and tail probability mixed with stop-loss, Eq. (34).

The key technical points of the derivation of the saddlepoint approximation for the tail probability have already been discussed. The derivation for the tail probability mixed with stop-loss proceeds through a step of contour rearrangements similar to Eq. (48), followed by the expansion in the vicinity of the saddlepoint and reduction to the standard integral (36).

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