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Options for collateral options

A *Risk* collateral management Cutting Edge feature, by Alexandre Antonov and Vladimir Piterbarg

Options for collateral options

When collateral can be posted in multiple currencies, pricing even the simplest derivatives involves optionality, which is often tackled numerically. But by conditioning on a risk factor to make variables independent, this can be simplified. Alexandre Antonov and Vladimir Piterbarg show this is both quicker and more accurate than more obvious methods

Credit support annexes specify rules for posting collateral. If multiple currencies are allowed, then the party posting collateral has, now and at each future point in time, a choice of which currency to post. This choice leads to optionality that needs to be accounted for when valuing even the most basic of derivatives, such as forwards or swaps (see Piterbarg, 2012).

In this article, we consider the important case of two currencies, under the assumption of full substitution rights (see Piterbarg, 2013). In this case, the adjustment to the discount factor applied to a cashflow paid at time T reduces to calculating an expression of the form:

$$D(T) \triangleq E\left(e^{-\int_0^T q(s)^+ ds}\right), T > 0 \qquad (1)$$

where the stochastic process $q(\cdot)$ represents the so-called collateral basis, that is, the difference between foreign exchange-adjusted collateral rates in the two currencies. Here and throughout we use the notation $x^+ \triangleq \max(x, 0)$. The collateral basis is typically modelled as a Gaussian process (see Piterbarg, 2012, and McCloud, 2013a), a choice we adopt here, too.

Even with the Gaussian assumption in place, the exact calculation of the expected value in (1) in closed form appears impossible. Jeanblanc, Pitman & Yor (1997) study similar functionals for a Brownian motion but the methods do not extend easily to a general Gaussian process, especially with variable parameters. However, a way to efficiently calculate $\{D(T_n)\}$ for a collection of times $\{T_n\}$ is of critical importance as they are needed for discounting of all collateralised over-the-counter derivatives.

To speed up calculations, a closed-form first-order approximation was proposed in Piterbarg (2012). This approximation appears to be sufficiently accurate for reasonable values of market parameters; however, under more stressed conditions its accuracy deteriorates. The aim of this article is to derive and test a number of more accurate, yet still numerically efficient, methods for approximating $\{D(T_n)\}$ as defined by (1), for a given set of times $0 < T_1 < \cdots < T_N = T_{\max}$.

Going beyond a first-order approximation, the obvious extension is to consider a second-order one. This can be done in different ways, such as calculating it as an exponential function of centred moments.

Better accuracy, however, is achieved by conditioning on a particular risk factor, so that the measurements of the collateral basis at different times become independent from each other, and each $D(T_n)$ becomes a product of expectations rather than an expectation of products. A Black-like formula can then be applied to calculate each term. The conditioning risk factor is obtained by matching the conditional and unconditional discount factors in a small volatility expansion. These methods are not only more accurate, but are also computationally more efficient for both typical and stressed market scenarios.

Model setup

We set the Gaussian collateral basis process to be of the form:

$$q(t) = f(t) + x(t) \qquad (2)$$

where $f(t)$ is a deterministic function and the process $x(t)$ satisfies the mean reverting stochastic differential equation:

$$dx(t) = -\chi(t)x(t)dt + \sigma(t)dW(t), \quad x(0) = 0 \qquad (3)$$

with mean reversion $\chi(t)$ and volatility $\sigma(t)$. The process $x(\cdot)$ can be rewritten as:

$$x(t) = h(t)y(t)$$

where:

$$h(t) = e^{-\int_0^t \chi(s)ds}$$

and we introduce the driftless Gaussian process:

$$y(t) = \int_0^t g(s)dW(s), \quad g(s) = \sigma(s)/h(s) \qquad (4)$$

First-order approximation

We start with a simple first-order approximation. For ease of notation, define:

$$I(T) \triangleq \int_0^T q(t)^+ dt$$

The idea behind the first-order approximation is given by the following calculation:

$$D(T) = E\big(\exp(-I(T))\big) \approx D_1(T) = \exp\big(-E(I(T))\big) \qquad (5)$$

In a typical application, the adjustment curve $D(T)$, $T \geq 0$, or its approximation $D_1(T)$, $T \geq 0$, is required for a large number of times T, to discount a potentially large derivatives portfolio. Normally one would choose a reasonably fine grid of, say, N times $\{T_n, n = 1, \ldots, N\}$ and calculate $D_1(T_n)$, $n = 1, \ldots, N$, for that grid. For T not in the grid, $D_1(T)$ is then obtained by an appropriate interpolation. For a suitably fine grid, we can write:

$$D_1(T_n) \approx \exp\left(-\sum_{k=1}^{n} E\left(q(T_k)^+\right)\Delta T_k\right), n = 1, \ldots, N$$

where $\Delta T_k = T_k - T_{k-1}$ and $T_0 = 0$. Therefore, to calculate all $D_1(T_n)$, $n = 1$,

$\dots, N$, we only need to calculate all b_n, $n = 1, \dots, N$, where we define:

$$b_n \triangleq E\left(q(T_n)^+\right)$$

As $q(T_n)$ is Gaussian by assumption, each b_n can be calculated in closed form by the application of the Bachelier formula:

$$b_n = E\left(q(T_n)^+\right) = m(T_n)\Phi\left(\frac{m(T_n)}{v(T_n)}\right) + v(T_n)\varphi\left(\frac{m(T_n)}{v(T_n)}\right)$$

with Φ and φ being the standard Gaussian cumulative distribution function and probability distribution function, respectively, and where:

$$m(t) = E(q(t)), \quad v(t)^2 = Var(q(t))$$

We consider calculation complexity in units of one valuation of the Bachelier formula. So it has cost $O(1)$, and the first-order approximation scheme for a time grid of size N has cost $O(N)$.

Second-order approximation

The first-order approximation may not be accurate enough for certain values of the parameters of the collateral basis process $q(\cdot)$. To improve the accuracy, a natural extension would be a second-order approximation. For the random variable $I(T)$, this leads to:

$$D(T) \approx D_2(T) \triangleq \exp\left(-E(I(T)) + \frac{1}{2}Var(I(T))\right) \tag{6}$$

Here:

$$Var(I(T)) = E\left(I(T)^2\right) - \left(E(I(T))\right)^2$$

where we have:

$$E\left(I(T)^2\right) = \int_0^T dt \int_0^T ds\, E\left(q(t)^+ q(s)^+\right)$$

The numerical scheme based on this direct extension to second order is significantly costlier because of the numerical integration in this last expression. To calculate $Var(I(T_n))$ for the time grid $\{T_n\}_{n=1}^N$, we would need to perform N^2 evaluations of the expression of the type $E(q(T_k)^+ q(T_n)^+)$:

$$\begin{aligned} &E\left(q(T_k)^+ q(T_n)^+\right) \\ &= \int_0^\infty x P\left(q(T_k) \in dx\right) E\left(q(T_n)^+ \middle| q(T_k) = x\right) \\ &\approx \sum_{l=1}^{L} x_l \varphi\left(\frac{x_l - m(T_k)}{v(T_k)}\right) E\left(q(T_n)^+ \middle| q(T_k) = x_l\right)\Delta x_l \end{aligned} \tag{7}$$

for a suitable discretisation $0 = x_0 < x_1 < \cdots < x_L < \infty$ with L space grid points. As shown in Antonov & Piterbarg (2013), the term $E(q(T_n)^+ | q(T_k) = x_l)$ is obtained by one evaluation of the Bachelier formula. The overall cost of this scheme is then $O(N^2L)$, which is significantly higher than that of the first-order approximation.

We also define another second-order approximation that will turn out to be more accurate in many cases:

$$\begin{aligned} \tilde{D}_2(T) &\triangleq \exp\left(-E(I(T))\right)\left(1 + \frac{1}{2}Var(I(T))\right) \\ &= D_1(T)\left(1 + \frac{1}{2}Var(I(T))\right) \end{aligned} \tag{8}$$

The overall cost for calculating $\tilde{D}_2$ is clearly the same as for D_2.

Conditional independence approach

General idea. The motivation for the conditional independence (CI) approach comes from the observation that, if all $q(t)$ were independent, then the calculation of the expected value in (1) would be decomposable into a product of expected values of the type $E(e^{-q(T_n)^+\Delta T_n})$, with each one calculated by a Black-like formula. Of course the $q(t)$ are generally not independent; however, by conditioning on a suitable factor we can make them approximately conditionally independent.

Let Z_0 be a standard, zero mean and unit variance Gaussian random variable and let $Q(t)$ be a zero mean Gaussian process, independent of Z_0 and such that $Q(t)$ is independent of $Q(t')$ for any $t \neq t'$.

For any function $\gamma(t)$, we can consider an approximation $\tilde{y}(t)$ to the process $y(t)$ underlying the rate (4) that is given by:

$$y(t) \approx \tilde{y}(t) \triangleq \gamma(t) Z_0 + Q(t) \tag{9}$$

We set the variance of the variable $Q(t)$ so that:

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$$Var(y(t)) = Var(\tilde{y}(t))$$

and so:

$$Var(Q(t)) = V(t) - \gamma(t)^2 \tag{10}$$

where $V(t)$ is the variance of $y(t)$:

$$V(t) \triangleq Var(y(t)) = \int_0^t g(s)^2 ds = \int_0^t e^{2\int_0^s \chi(u)du}\sigma(s)^2 ds \tag{11}$$

The approximation (9) is a one-factor approximation of the covariance matrix $E(y(t)y(t'))$:

$$E(y(t)y(t')) \approx \gamma(t)\gamma(t') + \left(V(t) - \gamma(t)^2\right)1_{\{t=t'\}} \tag{12}$$

The only requirement for the right-hand side of (12) to define a valid covariance matrix is that:

$$\gamma(t)^2 \le V(t), \quad t > 0 \tag{13}$$

We will see that our choices of $\gamma(\cdot)$ will satisfy this requirement.

Once the function $\gamma(\cdot)$ is chosen, and is approximated by (9), the discount factor adjustment in (1) is calculated by conditioning on Z_0 so that we can replace the expectation of the products by the product of expectations:

$$\begin{aligned} D(T_n) &= E\left(e^{-\int_0^{T_n} q(t)^+ dt}\right) \approx E\left(E\left(e^{-\sum_{i=0}^n q(T_i)^+\Delta T_i}\middle| Z_0\right)\right) \\ &\approx E\left(E\left(e^{-\sum_{i=0}^n \left(f(T_i)+\gamma(T_i)h(T_i)Z_0+h(T_i)Q(T_i)\right)^+\Delta T_i}\middle| Z_0\right)\right) \\ &\approx \hat{D}_{CI}(T_n) \end{aligned}$$

where:

$$\hat{D}_{CI}(T_n) \triangleq E\left(\hat{D}_{CI}(T_n, Z_0)\right) \tag{14}$$

$$\begin{aligned} \hat{D}_{CI}(T_n, z) &\triangleq \prod_{i=0}^n B_i(z) \\ B_i(z) &\triangleq E\left(e^{-\left(f(T_i)+\gamma(T_i)h(T_i)z+h(T_i)Q(T_i)\right)^+\Delta T_i}\right) \end{aligned} \tag{15}$$

Finally, we integrate over the conditioning variable Z_0 to obtain:

$$\hat{D}_{CI}(T_n) = \int\left(\prod_{i=1}^n B_i(z)\right)\varphi(z)dz, \quad n = 1,\ldots,N \tag{16}$$

Each term $B_i(z)$ is available in a closed form by one evaluation of the Black formula (cost $O(1)$):

$$B_i(z) = E\left(\max\left(e^{-\mu_i+\sigma_i(\beta_i z+\bar{\beta}_i Z_i)}, 1\right)\right)$$

where Z_i is $N(0, 1)$ Gaussian variable and:

$$\mu_i = \Delta T_i f(T_i) \tag{17}$$

$$\begin{aligned} \sigma_i &= \Delta T_i h(T_i)\sqrt{V(T_i)} \\ \beta_i &= \frac{\gamma(T_i)}{\sqrt{V(T_i)}}, \quad \bar{\beta}_i = \sqrt{1-\beta_i^2} \end{aligned} \tag{18}$$

As long as (13) is satisfied, the variance of $Q(\cdot)$ given by (10) is non-negative, and then $\beta_i \le 1$ and $\bar{\beta}_i$ are well defined.

Since the same term $B_i(z)$ appears in the calculation of all $D(T_n)$ for $n \ge i$, they can be reused and the cost of evaluating all $D(T_n)$, $n = 1, \ldots, N$ is given by $O(NL)$, with u being the size of the discretisation grid for the integral in (16).

The time-step convergence can be improved by a more careful choice of the means μ_i, as explained in Antonov & Piterbarg (2013). McCloud (2013b) in a similar context discretised the integrals in the right-hand side of the above equation as:

$$\int_{T_i}^{T_{i+1}} q(t)^+ dt \approx \left(\int_{T_i}^{T_{i+1}} q(t)dt\right)^+$$

and proposed a simple closed form for the upper bound of the discount factor adjustment in the case of the single-period discretisation.

Small volatility expansion. The choice of the function $\gamma(\cdot)$ is critical to the performance of the method, so criteria to determine the best $\gamma(t)$ given $V(t)$ are required. While the method is motivated by the conditional independence argument, clearly what we want is to match $\hat{D}_{CI}(t, z)$ to $D(t, z)$ as closely as possible, where:

$$D(T, z) \triangleq E\left(e^{-\int_0^T q(t)^+ dt}\middle| Z_0 = z\right)$$

So we pick $\gamma(\cdot)$ to minimise the error between the small volatility expansions of both $D(t, Z_0)$ and $\hat{D}_{CI}(t, Z_0)$.

As we will prove in Appendix A, the exact conditional discount factor can be expanded in small volatility as:

$$\begin{aligned} D(T,z) &= e^{-\int_0^T dt\left(f(t)+\gamma(t)h(t)z\right)^+} \\ &\times\Big\{1+\int_0^T dt'\tilde{\theta}(t',z)\int_0^{t'} dt\tilde{\theta}(t,z)\left(V(t)-\gamma(t)\gamma(t')\right) \\ &-\frac{1}{2}\int_0^T dt\delta\left(f(t)+\gamma(t)h(t)z\right)h(t)^2\left(V(t)-\gamma(t)^2\right)+O(\sigma^3)\Big\} \end{aligned} \tag{19}$$

where $\delta(\cdot)$ is the Dirac delta function and:

$$\tilde{\theta}(t,z) = \theta\left(f(t)+\gamma(t)h(t)z\right)h(t)$$

with $\theta(x) \triangleq 1_{\{x>0\}}$ the Heaviside function. A small volatility expansion of the conditionally independent discount factor in the limit $N \to \infty$ will read:

$$\begin{aligned} D_{CI}(T,z) &= e^{-\int_0^T dt\left(f(t)+\gamma(t)h(t)z\right)^+} \\ &\times\left(1-\frac{1}{2}\int_0^T dt\delta\left(f(t)+\gamma(t)h(t)z\right)h(t)^2\left(V(t)-\gamma(t)^2\right)+O(\sigma^3)\right) \end{aligned} \tag{20}$$

as shown in Appendix B. Comparing this expansion with (19), we notice that the difference comes from the term containing θ-functions. This means that the optimal choice of $\gamma(\cdot)$ should minimise the defect term:

$$\begin{aligned} R_+(T) = E\Big(& e^{-\int_0^T dt\left(f(t)+\gamma(t)h(t)Z_0\right)^+} \\ &\times\int_0^T dt'\tilde{\theta}(t',Z_0)\int_0^{t'} dt\tilde{\theta}(t,Z_0)\left(V(t)-\gamma(t)\gamma(t')\right)\Big) \end{aligned} \tag{21}$$

We can also define the simpler approximation to the defect without the '+' operator in the exponential:

$$R(T) = E\left(e^{-\int_0^T dt\left(f(t)+\gamma(t)h(t)Z_0\right)} \times \int_0^T dt' h(t') \int_0^{t'} dt\, h(t)\left(V(t)-\gamma(t)\gamma(t')\right)\right) \quad (22)$$

Formally, it corresponds to a limit of $f(t) \to +\infty$ and is related to the variance of the integral $I(\cdot)$ without the '+' operator. We will choose $\gamma(\cdot)$ to minimise either of these functions.

Variance fit. We start with a simpler approach of minimising the approximate defect $R(T)$ in (22) rather than $R_+(T)$ in (21). We consider the following integral equation for $\gamma(\cdot)$, $R(T) = 0$, $T \geq 0$, which translates into:

$$\int_0^T dt'\, h(t')\gamma(t') \int_0^{t'} dt\, h(t)\gamma(t) = \int_0^T dt'\, h(t') \int_0^{t'} dt\, h(t)V(t), \quad T \geq 0$$

This can be simplified to:

$$\left(\int_0^T dt\, h(t)\gamma(t)\right)^2 = 2\int_0^T dt' h(t') \int_0^{t'} dt\, h(t)V(t)$$

We recognise the right-hand side to be $Var(\int_0^T x(t)dt)$, which leads us to the solution:

$$\gamma(T) = \frac{1}{h(T)} \frac{d}{dT}\left(Var\left(\int_0^T x(t)dt\right)\right)^{1/2}, \quad T \geq 0 \quad (23)$$

It is not hard to see that with this choice of $\gamma(\cdot)$, we have that:

$$Var\left(\int_0^T q(t)dt\right) = Var\left(\int_0^T h(t)\left(\gamma(t)Z_0 + Q(t)\right)dt\right), \quad T \geq 0 \quad (24)$$

and in particular:

$$E\left(e^{\int_0^T q(t)dt}\right) = E\left(e^{\int_0^T \left(f(t)+h(t)\left(\gamma(t)Z_0+Q(t)\right)\right)dt}\right), \quad T \geq 0 \quad (25)$$

So this $\gamma(\cdot)$ allows us to match the simplified adjustment curve to the one defined by the conditionally independent process $Q(\cdot)$ (also without the '+' operator), before taking the small volatility expansion. For that reason we call this the variance fit (VF) version of the CI approach.

We show in Appendix C that $\gamma(\cdot)$ as defined in this section satisfies the restriction (13).

Optimal fit. To improve on the method above, we focus on the original defect $R_+(T)$, $T \geq 0$. The zero defect (21) condition is approximately equivalent to:

$$0 \approx E\left(\int_0^T dt'\tilde{\theta}(t', Z_0) \int_0^{t'} dt\tilde{\theta}(t, Z_0)\left(V(t) - \gamma(t)\gamma(t')\right)\right) \quad (26)$$

Differentiating with respect to T, this simplifies to:

$$0 \approx E\Big(\theta\left(f(T)+\gamma(T)h(T)Z_0\right) \int_0^T \theta\left(f(t)+\gamma(t)h(t)Z_0\right)h(t)\left(V(t)-\gamma(t)\gamma(T)\right)dt\Big)$$

Now, taking the expected value of the theta functions, we obtain the system of equations:

$$0 \approx \int_0^T \Phi\left(\min\left(\frac{f(T)}{\gamma(T)h(T)}, \frac{f(t)}{\gamma(t)h(t)}\right)\right)h(t)\left(V(t)-\gamma(t)\gamma(T)\right)dt,$$
$$T \geq 0$$

These equations can be solved numerically. Having a discretised time grid $\{T_n\}_{n=0}^N$, and having calculated $\gamma(T_0)$, ... , $\gamma(T_{n-1})$, we can find $\gamma(T_n)$ by numerically solving the one-dimensional equation above.

To further speed up the calculations without sacrificing much accuracy, we use γ_0 obtained by the formula (23) inside the Gaussian cumulative distribution function instead of γ. This gives us:

$$\gamma(T) = \frac{\int_0^T dt\Phi\left(\min\left(\frac{f(T)}{\gamma_0(T)h(T)}, \frac{f(t)}{\gamma_0(t)h(t)}\right)\right)h(t)V(t)}{\int_0^T dt\Phi\left(\min\left(\frac{f(T)}{\gamma_0(T)h(T)}, \frac{f(t)}{\gamma_0(t)h(t)}\right)\right)h(t)\gamma(t)}, \quad T \geq 0 \quad (27)$$

This equation can be iterated forward to go from $\gamma(T_0)$, ... , $\gamma(T_{n-1})$ to $\gamma(T_n)$ without the need for a numerical solver. The formula (27) defines the optimal fit (OF) version of the CI approach.

While we cannot prove that γ given by this formula satisfies (13) by construction, numerical experiments show that this appears to be the case. To make sure the algorithm always works, we suggest simply capping $\gamma(T)$ from (27) with $\sqrt{V(T)}$ on each step of the algorithm.

A. Timing results

Method	N_x	N	Time (ms)
FD	3,520	140	81.2
CI OF	20	35	4.9
CI VF	20	25	0.3
SO/SO exp	20	25	2.2

B. Typical setup

T	Intrinsic rate	Difference with intrinsic rate (bp)				
		FD	CI OF	CI VF	SO	SO exp
1	0.0	0.5	0.1	0.5	0.4	0.4
5	0.0	4.8	4.4	4.7	4.6	4.6
10	0.0	8.9	8.4	8.5	8.5	8.5
15	0.0	13.5	12.8	12.9	12.9	12.9
20	0.0	19.2	18.3	18.6	18.5	18.5
30	12.5	22.5	21.1	21.8	21.5	21.5
40	37.4	19.1	17.3	18.4	18.0	18.0

C. Stressed setup

T	Intrinsic rate	Difference with intrinsic rate (bp)				
		FD	CI OF	CI VF	SO	SO exp
1	0.0	48.8	48.5	48.6	48.6	48.6
5	0.0	139.4	139.7	140.2	138.8	138.8
10	0.0	178.9	179.8	180.9	175.0	174.5
15	0.0	195.4	197.0	198.7	186.4	184.0
20	0.0	204.4	206.8	209.1	190.8	184.6
30	12.5	203.8	207.9	211.8	185.8	167.5
40	37.4	189.3	195.0	201.2	173.4	139.1

APPENDIX A: EXACT FORMULA EXPANSION

Here, we prove the expansion formula (19). Suppose we have a zero mean process $u(t)$, with $u(0) = 0$. Given a deterministic level $l(t)$, we can expand the following average around a small ε:

$$E\left(e^{-\int_0^T (l(t)+\varepsilon h(t)u(t))^+ dt}\right) \approx e^{-\int_0^T l(t)^+ dt}\left(1-\varepsilon E\left(\int_0^T \theta(l(t))h(t)u(t)dt\right)\right.$$
$$\left.+\frac{\varepsilon^2}{2}E\left(\left(\int_0^T \theta(l(t))h(t)u(t)dt\right)^2\right)-\frac{\varepsilon^2}{2}E\left(\int_0^T \delta(l(t))h(t)^2 u(t)^2 dt\right)\right)$$

Because u has zero mean and setting ε to one:

$$\frac{E\left(e^{-\int_0^T (l(t)+h(t)u(t))^+ dt}\right)}{e^{-\int_0^T l(t)^+ dt}}$$
$$\approx 1+\int_0^T dt'\theta(l(t'))h(t')\int_0^{t'} dt\theta(l(t))h(t)E(u(t)u(t'))$$
$$-\frac{1}{2}\int_0^T \delta(l(t))h(t)^2 E\left(u(t)^2\right)dt$$

The expansion (19) follows from this result provided that we identify $l(t)$ with $f(t) + \gamma(t)h(t)z$ and $E(u(t)u(t'))$ with $V(\min(t, t')) - \gamma(t)\gamma(t')$. This is legitimate because of the approximate independence of the residual process $y(t) - \gamma(t)Z_0$ from Z_0 coming from (9). Then, conditional on $Z_0 = z$, the process $y(t)$ will have the following characteristics:

$$E(y(t)|Z_0 = z) \approx \gamma(t)z$$

and:

$$E(y(t)y(t')|Z_0 = z) \approx V(\min(t,t')) - \gamma(t)\gamma(t')$$

which finishes the proof.

Test results

We present test results in terms of the effective rates, $r(T) = -T^{-1}\ln D(T)$ for different calculation methods:

- FD – the finite difference method (details can be found in Antonov & Piterbarg, 2013).
- CI OF – conditional independence method (16) with the optimal fit (27).
- CI VF – conditional independence method (16) with the variance fit (23).
- SO – the second-order approximation defined by (8).
- SO exp – the second-order approximations defined by (6).

We present the effective rates and approximations $r(T)$ for $T = 1, 5, 10, 15, 20, 30, 40$ years. We consider collateral choice options of different moneyness as modelled by the function $f(t)$ in (2), which we take to be linear with start $f(0) = -1.5\%$ and end $f(40) = 1.5\%$ points. According to our experiments, this corresponds to a fairly challenging case. Numerical results for other values of $f(0)$ and $f(40)$ can be found in Antonov & Piterbarg (2013).

We look at two regimes for the process $x(t)$. The first typical market scenario setup uses (time-independent) volatility and mean-reversion $\sigma(t) = 0.01$ and $\chi(t) = 0.4$. The second stressed scenario setup uses $\sigma(t) = 0.04$ and $\chi(t) = 0.1$.

The performance of each method depends on the number of time (N) and space (N_x) steps. Each method converges to a limit as N and N_x become large. The limits, of course, are different for different methods due to biases introduced by their approximation assumptions. The FD method is bias-free in the limit. In the test results shown below, for each method we used N and N_x such that the largest absolute error over all maturities against the appropriate limit is less than 0.5 basis points. In table A, we show these N, N_x and time in milliseconds for calculating rates for all maturities to 40 years for the typical parameter setup, on a standard workstation.

There are significant speed advantages of analytical methods with respect to the finite-difference scheme, with the CI VF method being a clear winner. As we shall see, the CI methods are the most accurate.

In the tables where we present results on accuracy, we show the intrinsic rate value defined as $r_I(T) = T^{-1}\int_0^T(E(q(t)))^+dt = T^{-1}\int_0^T f(t)^+dt$ and a difference of corresponding rates with it, all expressed in basis points. Calculations go up to 40 years.

We observe that for the typical setup in table B all approximation methods perform very well, with the error with respect to the benchmark FD method not exceeding 1–2bp. The picture changes for the stressed setup in table C,

APPENDIX B: CI APPROXIMATION EXPANSION

Here, we derive the expansion formula (20) for the conditional discount factor. As in the previous section, we have:

$$E\left(e^{-\Delta T_i\left(l(T_i)+\varepsilon h(T_i)Q(T_i)\right)^+}\middle|Z_0\right)$$
$$\approx e^{-\Delta T_i\left(l(T_i)\right)^+}\times\left(1-\varepsilon E\left(\Delta T_i\theta\left(l(T_i)\right)h(T_i)Q(T_i)\middle|Z_0\right)\right.$$
$$+\frac{1}{2}\varepsilon^2E\left(\left(\Delta T_i\theta\left(l(T_i)\right)h(T_i)Q(T_i)\right)^2\middle|Z_0\right)$$
$$\left.-\frac{1}{2}\varepsilon^2E\left(\Delta T_i\delta\left(l(T_i)\right)h(T_i)^2Q(T_i)^2\middle|Z_0\right)\right)$$

where $l(T_i) = f(T_i) + \gamma(T_i)h(T_i)z$. As $Q(t)$ has mean zero, combining all the terms in the discount factor (15) gives:

$$\hat{D}_{CI}(T_n,z)\simeq\prod_{i=0}^{N}e^{-\Delta T_i\left(l(T_i)\right)^+}$$
$$\times\prod_{i=0}^{N}\left(1+\frac{1}{2}(\Delta T_i)^2\theta\left(l(T_i)\right)h(T_i)E\left(Q(T_i)^2\right)\right.$$
$$\left.-\frac{1}{2}\Delta T_i\delta\left(l(T_i)\right)h(T_i)E\left(Q(T_i)^2\right)\right)$$

In the continuous limit, $\Delta T_i \to 0$, we only retain terms of order $O(\Delta T_i)$ and obtain:

$$D_{CI}(T,z)=e^{-\int_0^T\left(f(t)+\gamma(t)h(t)z\right)^+dt}$$
$$\times\left(1-\frac{1}{2}\int_0^T\delta\left(f(t)+\gamma(t)h(t)z\right)h(t)^2E\left(Q(t)^2\right)dt\right)$$

leading to (20).

where errors become much larger.

We see that the CI with the optimal fit outperforms all other methods in accuracy. The CI VF also has good accuracy, while the second-order expansions (especially the exponential form of it) have rather poor precision.

Conclusion

Collateral choice options complicate derivatives pricing, and make numerical methods essential. Finite difference approximations, first- and second-order Taylor expansions, and conditionally independent approximations have varying characteristics in terms of computing speed and accuracy. But analytic methods have the best results, with two varieties of conditional independence approximations leading the way in numerical tests of typical and stressed scenarios.

Further work should examine the extension of these and other methods to multicurrency collateral options, which is not straightforward due to the slowness of the numerical integration in multiple dimensions. **R**

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APPENDIX C: VF CI: $\gamma(T)$ UPPER BOUNDARY

Here, we prove that:

$$\gamma(t)^2\le V(t),\quad t\ge0 \tag{28}$$

for the function $\gamma(\cdot)$ defined for the variance fit CI procedure by (23), or, equivalently, the solution to the equation:

$$\int_0^T dt'h(t')\int_0^{t'}dt\,h(t)\left(V(t)-\gamma(t)\gamma(t')\right)=0 \tag{29}$$

For small t, the function $h(t)$ is approximately constant, $h(t)\approx1$. Therefore, (29) can be rewritten as:

$$\left(\int_0^T dt\gamma(t)\right)^2=2\int_0^T dt'\int_0^{t'}dt\,V(t)$$

Moreover, for small t, the variance grows linearly, $V(t)\approx Ct$. The equation above can then be solved, for small T, to yield:

$$\gamma(t)\approx\frac{\sqrt{3}}{2}\sqrt{V(t)}$$

Thus, $\gamma(t)$ satisfies (28) for small t and, in particular, for $t = 0$.

Differentiating (29) with respect to T, we obtain:

$$\gamma(T)=\frac{\int_0^T dt\,h(t)V(t)}{\int_0^T dt\,h(t)\gamma(t)}$$

and differentiating again, we obtain the following ODE:

$$\gamma'(t)=\frac{h(t)}{\int_0^t dt'h(t')V(t')}\gamma(t)\left(V(t)-\gamma(t)^2\right) \tag{30}$$

From this we see that the derivative $\gamma'(t)$ tends to zero when $\gamma(t)$ approaches $\sqrt{V(T)}$. As the initial time derivative value $\gamma'(0)$ is positive, in line with the small time discussion above, we see that $\gamma'(t)\ge0$ for all $t\ge0$ and it follows from (30) then that (28) is always satisfied.

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