

Risk Neutral Modeling for Economic Scenario Generation: In Theory and Practice

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Abstract

This article explores the theory behind and practice surrounding Risk Neutral Modeling for Economic Scenario Generation (ESG). In addition to laying out the foundations of the Risk Neutral Measure and Fundamental Theorem, this study also sets forth several practical case study examples that demonstrate the importance of risk neutral modeling and joint calibration in market consistent valuation, regulatory reporting, risk and capital forecasting—and the overall enhancement of a firm’s risk management.

The volatile markets and evolving regulatory climate of recent years have driven much attention to risk management in the insurance industry. With many insurance companies subject to more complex regulations, practitioners are seeking to gain a wider picture of risk exposures—especially as their firms adopt more complex investment-linked Insurance product offerings. Overall, and not surprisingly, the industry is seeing a movement away from the purely deterministic approaches that may have been used in the past.

A new trend on the horizon is the use of sophisticated stochastic simulation frameworks that can be used to produce risk neutral and real world economic scenarios. Furthermore, scenario sets can now be easily integrated with in-house or other third party liability systems, providing model consistency across the firm.

Using more precise calibrations of multi-factor models to reduce model risk is becoming more prevalent. It is now possible to specify component models and correlation, and then apply a joint calibration process to accurately capture the observed market dynamics of each risk factor while correlating the risk factors. Unified hybrid models are playing a key role in capturing correlation among economies and asset classes.

Essentially, risk management has assumed a new life of its own when it comes to today’s insurance industry. Whether fully immersed in ESG, or just beginning to consider the implementation of ESG techniques, this article highlights the importance of skillfully leveraging a solid and consistent modeling and joint calibration framework to achieve effective risk management in today’s ever-increasingly complex insurance marketplace.

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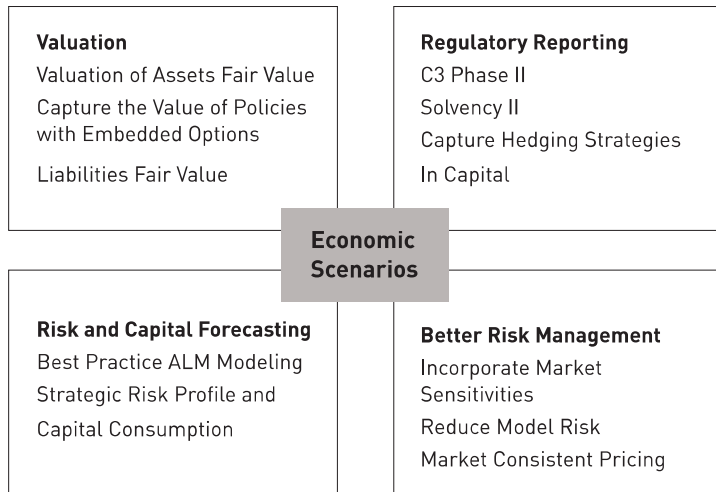
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I. BACKGROUND: WHY DO INSURANCE COMPANIES NEED AN ESG?



What are Economic Scenarios/Economic Scenario Generators?

An example of a typical workflow would be an ESG engine that generates multiple scenarios reflecting the risk factors across various asset classes, tied to one or multiple currencies. Modeled indices are used as proxies to model the assets backing the liabilities, backing the free surplus or simply representing the separate accounts a policy may be invested in. An Asset Liability Management (ALM) Engine can range from a simple spreadsheet, to a full actuarial projection software. For example, interest rate risk may require modeling the full yield curve at future time steps, in order to reflect future discounts to liability cashflows. In this case, the economic scenarios should capture zero coupon bond prices for several tenors at each time step and for each scenario, as seen in the example below:

Interest Rate Risk >

| Scenarios | t= | 0 | 1 | 2 | ... | 48 | 49 | 50 |
|---------------|---------|----------|----------|----------|-----|----------|----------|----------|
| Scenario 1 | ZCB 1 | 0.999297 | 0.999297 | 0.999297 | ... | 0.998509 | 0.998728 | 0.998661 |
| | ZCB 2 | 0.999252 | 0.999222 | 0.999277 | ... | 0.998423 | 0.998748 | 0.998567 |
| | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| | ZCB 600 | 0.103746 | 0.151337 | 0.166203 | ... | 0.129762 | 0.15942 | 0.135007 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | |
| Scenario 1000 | ZCB 1 | 0.998612 | 0.998549 | 0.998633 | ... | 0.998875 | 0.998875 | 0.998875 |
| | ZCB 2 | 0.998434 | 0.998553 | 0.998615 | ... | 0.998684 | 0.998899 | 0.998835 |
| | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| | ZCB 600 | 0.103746 | 0.151337 | 0.166203 | ... | 0.129762 | 0.15942 | 0.135007 |

II. THE RISK NEUTRAL MEASURE AND THE FUNDAMENTAL THEOREM

Next, to provide a background understanding for the case studies that follow, we will review the fundamental principles behind the Asset Pricing Theory, the Risk Neutral Measure and no-arbitrage principle.

A. **Asset Pricing Theory** is a strong and rigorous mathematical theory that sometimes relies on strong theoretical assumptions. The fundamentals of the theory are used by practitioners for derivative pricing and hedging in a “real world” environment. When using this theory, we assume two types of assets:

- 1- Risk Free Asset: Has a known return and bears no risk (government bonds ‘used to be’ a good example)
- 2- Risky Asset: The return is not known in advance (Stocks, Corporate Bonds etc.)

B. The Fundamental Theorem (under a set of mathematical assumptions):

Non-Arbitrage assumptions is equivalent to the existence of a **martingale measure**.

(Note: The martingale measures are also called risk neutral measures)

Furthermore, under additional mathematical assumptions (market completeness) we have a mathematical definition of the price of any option with a certain payoff, which is equal to the expected value of the discounted payoff under the martingale (risk neutral) measure.

C. What is the Non-Arbitrage Assumption?

Arbitrage is the ability of building a zero cost strategy that has a non-null probability of realizing strictly positive gains and no potential loss. Under the **Non-Arbitrage Assumption**, such a strategy does not exist.

D. What is a Risk Neutral Measure?

A Risk Neutral Measure (or martingale measure) is a measure of the event space that makes all discounted (following the risk free asset) assets martingales:

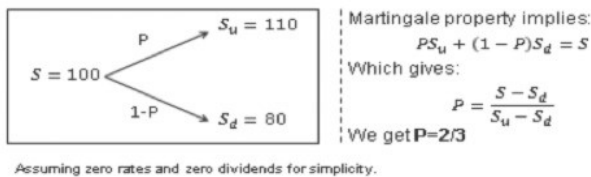
Expected Value Under "Risk Free" Measure $\rightarrow E_t[B_{t+1}S_{t+1}] = B_tS_t$

In particular: $E_0[B_1S_1] = S_0$

Where the process B_t is called the discounting process and represent the value of 1 dollar cash flow at time t seen from time 0.



Below represents an example of discrete time model with two states view of potential outcomes in an economy, S is a risky asset (an Equity, for example), where the rates and dividends are zero. We define the two potential outcomes being S_u and S_d and leave the probability of the two outcomes undefined. This probability will be the key to defining the measure:



Note that we didn't consider any derivative in this case. The dynamic of the underlying S is solely determined by the probability P , which in turn, is defined through the martingale property.

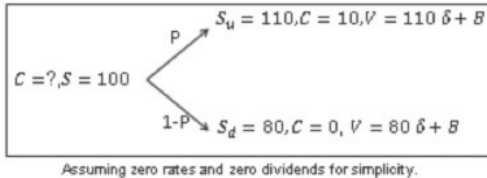
Case Example 1: The Non-Arbitrage Price

We can observe that this case considers a call with strike 100, maturing at the time step 1 under the same simplifying assumptions. The non-arbitrage price of this call will be calculated based on a replicating portfolio. This portfolio will be

constrained to have the same outcome (the pay-off of the call option), regardless of the state of the world and through a non-arbitrage argument, the price of the call will be determined:

$$V = \delta S + B \begin{cases} \text{Holding } \delta \text{ in the Asset } S; \\ \text{Holding } B \text{ in cash;} \end{cases}$$

We let the portfolio evolve without adding or removing any cash: which is essentially, a Self-Financing Portfolio.



Solving for $V=C$ on both Up and Down scenarios, we get $\delta=1/3$ and $B=-80/3$. Since this is a replicating portfolio, by **non-arbitrage** its value at zero should equal the price of the call (the proof by absurd is easy to establish through the creation of a portfolio that represents an arbitrage, if the price of the call is different than the replicating portfolio's value).

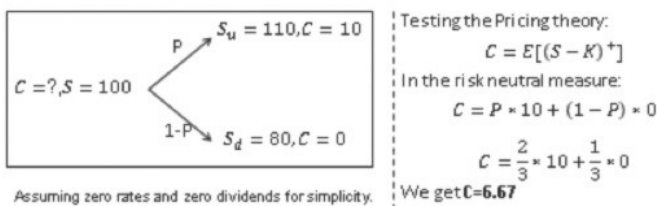
$$C = \delta S + B$$

Therefore, **C=6.67**.

Case Example 2: The Price of a Call Under Risk Neutral

Now, what is the price of the same option under the Risk Neutral Measure? We are in fact testing the pricing theory to see if we get the same price as we did using the no-arbitrage argument. The theory says that the price is the expected discounted value of the payoff:

$$C = E_t [B_t (S - K)^+]$$



Therefore, we do indeed get the same price using the risk neutral theory: **C=6.67**.

In summary, we observed that by relying on pure economic consideration based on non-arbitrage, we found the price of the call option and found the same price using the asset pricing theory formula in a risk neutral measure.

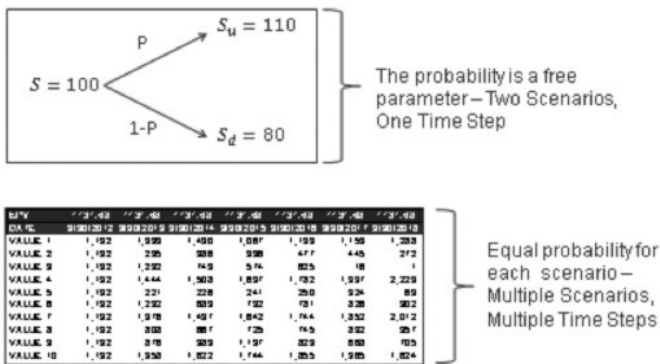
Then, using the risk neutral theory, we found a mathematical framework to find the non-arbitrage price of the same security. The risk neutral provides the mathematical framework to calculate the price and find the replicating portfolio. This replicating portfolio, although induced in a risk neutral world, will still be a replicating portfolio, when diffusing the underlying in the real world environment.

However, a different P would have given us the wrong call option value. And, when we expand this to the calculation of greeks, we can see how risk neutral is important to find the appropriate replicating portfolio that would guarantee, mathematically, a good hedge and therefore would be the best candidate for real world hedging programs and market consistent valuation applications.

What About the Economic Scenarios?

Comparing the Example to the Case of Economic Scenarios

Here, we gain the ability to use a model to come up with the prices. The free parameters don't lie into determining the probability of each scenario, but rather how many scenarios in a certain range. Said differently, we are targeting a distribution through a discrete representation of the different states of the economy for various time steps.



* Above, we should note that the values of the equity must be determined in such a way that the Expected Present Value is equal to the Spot Value.

| EPV | 1131.68 | 1131.68 | 1131.68 | 1131.68 | 1131.68 | 1131.68 | 1131.68 |
|----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| DATE | 3/30/2012 | 3/30/2013 | 3/30/2014 | 3/30/2015 | 3/30/2016 | 3/30/2017 | 3/30/2018 |
| VALUE 1 | 1,132 | 1,399 | 1,430 | 1,067 | 1,133 | 1,159 | 1,288 |
| VALUE 2 | 1,132 | 235 | 386 | 396 | 477 | 445 | 272 |
| VALUE 3 | 1,132 | 1,232 | 743 | 574 | 625 | 16 | 1 |
| VALUE 4 | 1,132 | 1,444 | 1,508 | 1,637 | 1,782 | 1,997 | 2,223 |
| VALUE 5 | 1,132 | 221 | 226 | 241 | 250 | 324 | 69 |
| VALUE 6 | 1,132 | 1,292 | 689 | 732 | 781 | 826 | 902 |
| VALUE 7 | 1,132 | 1,376 | 1,437 | 1,642 | 1,744 | 1,852 | 2,012 |
| VALUE 8 | 1,132 | 808 | 667 | 725 | 745 | 832 | 957 |
| VALUE 9 | 1,132 | 876 | 989 | 1,197 | 829 | 668 | 705 |
| VALUE 10 | 1,132 | 1,358 | 1,622 | 1,744 | 1,855 | 1,965 | 1,624 |

(Note: There is an equal probability for each scenario above.)

In conclusion, after observing the earlier examples, we can see that under Risk Neutral, the scenarios need to satisfy the martingale property, regardless of the number of paths. A normalization approach (at the model level) can guarantee that the measure stays perfectly Risk Neutral and will, therefore, guarantee the Non-Arbitrage Price. This approach avoids convergence issues of the martingale properties. As seen in the two states of the world example, any divergence of the martingale property on the underlying S lead to a similar divergence of the Risk Neutral Price with regards to the no-arbitrage price. However, practitioners need to be mindful that a non-perfect Risk Neutral measure will lead to the "mispricing" of the optionality of the liabilities with regards to the option price captured in the original market model.

III. MARKET CONSISTENCY AND JOINT CALIBRATION

A. Model Choice and Joint Calibration

As we continue our study, the Risk Neutral Framework will be used for market consistent valuation, which can ultimately be used for the calculation of replicating the portfolio of a set of sensitivities (greeks). We have observed the economic scenarios used to replicate the current market conditions for the purpose of market consistency valuation. In other terms, the models used to generate the scenarios have to be calibrated to the spot market values:

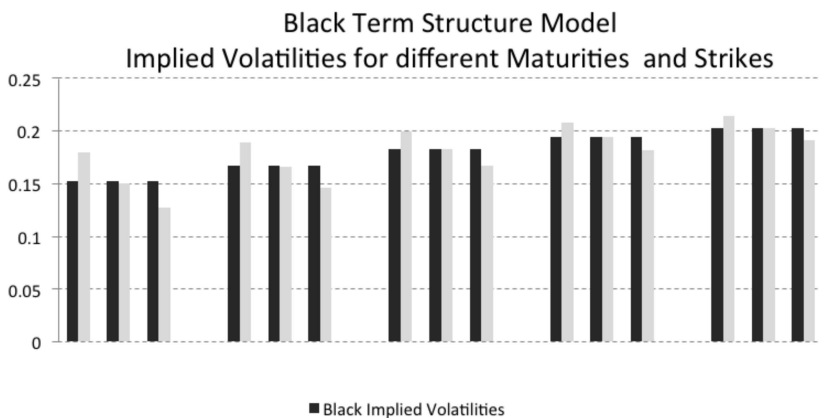
Example of an Implied Volatility Surface of a Major Equity Index

| | | | | | | | | | |
|-------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| | 80.0% | 90.0% | 95.0% | 97.5% | 100.0% | 102.5% | 105.0% | 110.0% | 120.0% |
| 1210 @ STX. | 1981.8 | 2229.6 | 2353.4 | 2415.3 | 2477.3 | 2539.2 | 2601.1 | 2725.0 | 2972.7 |
| 30D | 39.04% | 27.68% | 23.48% | 21.97% | 18.96% | 16.79% | 16.60% | 17.58% | 24.57% |
| 60D | 32.68% | 25.67% | 22.92% | 21.84% | 19.48% | 17.85% | 17.20% | 16.85% | 18.68% |
| 3M | 31.34% | 25.46% | 23.00% | 22.13% | 19.81% | 18.68% | 17.97% | 17.14% | 17.79% |
| 6M | 29.71% | 25.41% | 23.47% | 22.63% | 21.31% | 20.42% | 19.69% | 18.47% | 17.16% |
| 12M | 28.34% | 25.18% | 23.72% | 22.98% | 22.60% | 22.02% | 21.43% | 20.34% | 18.69% |
| 18M | 27.20% | 24.37% | 23.05% | 22.14% | 23.35% | 23.15% | 22.63% | 21.62% | 20.02% |
| 24M | 26.27% | 23.52% | 22.23% | 21.05% | 24.16% | 24.36% | 23.87% | 22.87% | 21.28% |

In the following example, we will consider a relatively complex liability that is sensitive to both interest rates and equities in a path dependent manner—in addition to deterministic actuarial assumptions for decrements. For simplicity, we will consider one Economy and a single Equity Index. In order to capture the current market price of the embedded options, the model used to generate scenarios must reproduce the market prices. The prices are quoted in terms of implied volatilities (*not to be confused with the realized or statistical volatility*). Long-dated option prices are required when the liability is of a long duration, which can bring the issue of the reliability of the actual market prices, when there is not enough liquidity in a certain market to sufficiently define a price. We should also keep in mind that the choice of model and the calibration approach is of key importance during this process. We will not discuss the relevance of market prices, but rather the appropriate choice of market model to capture these prices.

The first example shows the difference between market prices and model prices when choosing an improved version of the traditional Black Scholes model calibrated to the same option prices. The improvement lies in the ability to define a time dependent deterministic volatility (the volatility is still deterministic in this case).

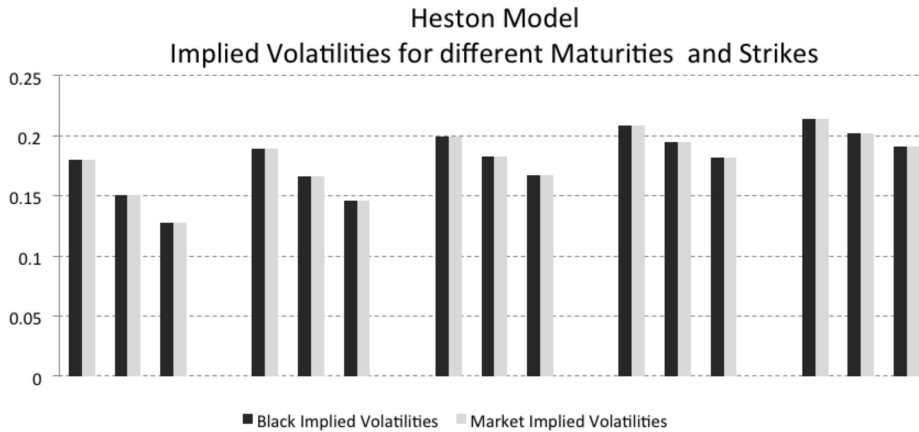
Example 1: Market Consistency for a Single Equity Model, the Black Scholes Term Structure Case



Above, we can observe the skew effect in the market prices (light gray columns), and we can also observe that although ATM option prices produced by Black Scholes term structure fit with the ATM market prices, the model fails to reproduce out-of-the-money and in-the-money market prices. This limitation in the Black Scholes model has been widely recognized and exposed in the literature.

The next example explores the quality of fit of a Heston model, which is a log normal model, similar to Black Scholes, but incorporates stochastic volatility—which happens to add more flexibility in the replication of the implied volatility structure. The following chart shows the differences between prices observed in the market and prices reproduced by Heston model after calibrating the model to the same option prices.

Example 2: Market Consistency for a Single Equity Model, the Heston Stochastic Volatility Case

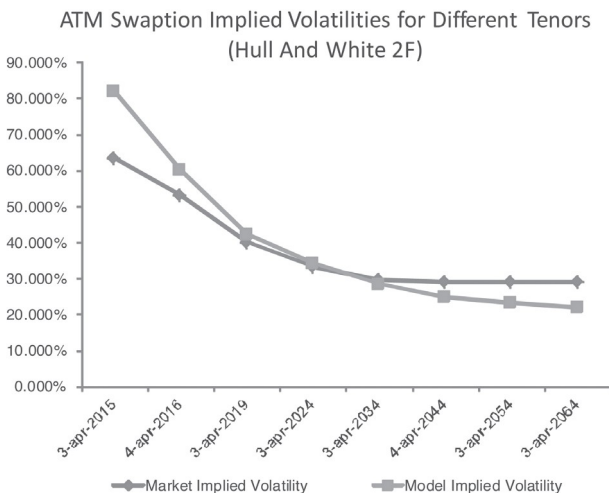


The Heston model exhibits a superior quality of fit, and will therefore be more adequate in terms of capturing market consistent prices in Economic Scenarios. It is worth noting that Heston model has also statistical properties for the generated underlying distributions that are closer to the historical behavior of equity markets (e.g. Fat tails).

The next two examples demonstrate the same logic for two different choices of Interest Rate models. The calibration for Interest Rate model is performed on multiple at-the-money Swaptions, spanning through multiple maturities and multiple tenors. The Swaption instrument selection should cover the length of the intended projection in order to capture the interest rate volatility for the entire projection.

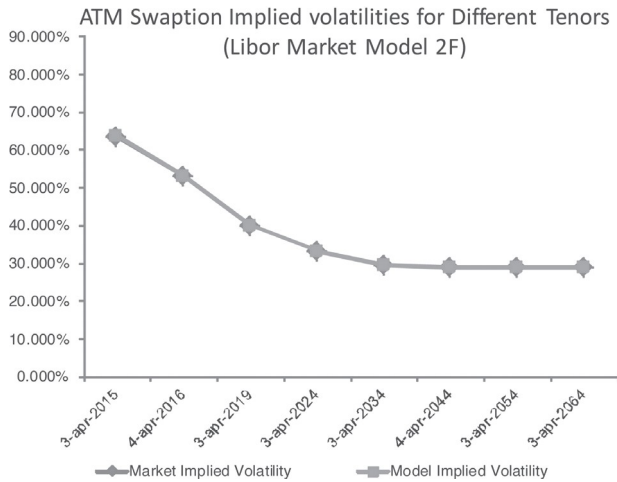
Example 3: Hull and White Two Factor Calibration to ATM Swaptions

This first Interest Rates example shows the goodness of fit of Hull and White 2 factors. Although it has some skew effect, it fails to replicate the shape observed in the market.



Example 3: Libor Market Model Two Factor Calibration

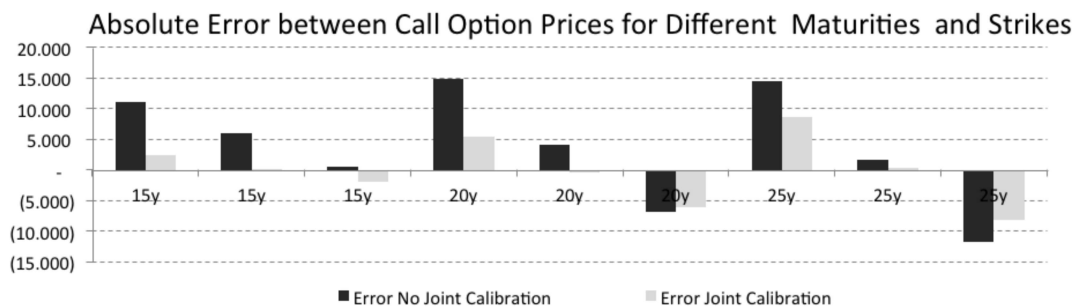
This second Interest Rates example shows the goodness of fit of Libor Market Model two factors. It has a clear superior fitting capability to the market implied volatilities. The market and the model implied volatilities curves overlap in this graph.



The Libor Market Model will, therefore, be more adequate in terms of capturing market consistent prices in Economic Scenarios.

Hybrid Joint Calibration

The two models described previously can be combined (given an Interest Rate – Equity correlation structure), into the so-called Hybrid model. When using Hybrid model, recalibration has to happen in order to take into account the effect of stochastic rates on the equity option prices. This effect is accentuated when looking at long dated options. This Hybrid model calibration is what is referred to as Joint calibration. The following chart shows the differences between market prices and Hybrid model prices when looking at long durations with and without joint calibration. It is worth noting that the same perfectly calibrated standalone models mentioned in the previous sections are used in this example (Heston and Libor Market Model).



Joint calibration significantly improves the goodness of fit of the overall Hybrid framework.

B. Market Consistency and Joint Calibration: A Case study on a GMWB for Life

The previous sections showed the clear impact of model setups and joint calibration capabilities into the calibration results and goodness of fit, and therefore, each setup would be more or less market consistent. What is the impact on the calculation of the optionality of a complex liability? Let's consider a GLWB (5 for life) for an age 55, with 10 years accumulation period, Annual Guarantee step-up and static lapse assumption and deterministic mortality:

| NAME | VALUE |
|---------------------------|----------|
| Starting Guaranteed Value | \$100.00 |
| Starting Account Value | \$100.00 |
| Withdrawal Rate | 5.00% |
| Fee Rate | 0.15% |

Calculating the present values of claims and comparing it to the present values of fees, gives different results depending on the model setup. *The better the calibration of the model setup is—the closer the numbers are to the market value of the policy.*

| | PV Fees | PV Claims |
|---------------------------------------|---------|-----------|
| LMM2F + BLACK SCHOLES | 1.746 | 1.749 |
| LMM2F + HESTON with Joint Calibration | 1.64 | 2.12 |
| LMM2F + HESTON No Joint Calibration | 1.69 | 1.74 |

Each set-up leads to different Present Values. Above, we can see the significant impact of joint calibration on PV Claims, and the importance of capturing the long-dated optionality. From a market consistent perspective, the more preferred model set-up is the second set-up LMM2F + HESTON (with joint calibration), which best captures more of the optionality of this GLWB since it is the most market consistent model according to the previous section. If we priced the same policy using the same combination of models without joint calibration, we would have gotten a PV of Claims equal to \$1.74. Recalling that without joint calibration, we are not capturing close market prices of long dated options, we are underestimating the richness of the guarantee compared to where the market is. In comparison, performing an appropriate joint calibration with the appropriate models shows a higher PV of claims (\$2.12), which means that the guarantee is richer according to the market. If a simple model for equities (not market consistent outside-of-the-money) is used, we can perceive that the PV of fees is very close to the PV of claims as seen in the first model setup (LMM2F and Black Scholes), and which doesn't reflect the market value of options.

CONCLUSION: MARKET CONSISTENCY AND JOINT CALIBRATION

As we have observed, in order to capture the optionality on prices, the choice and calibration of the model should lead to a model price consistent with the market price. A good combination of models in a given market environment will lead to a good calibration. The optionality embedded in the liability can be reflected through the use of Risk Neutral scenarios generated with the appropriate models. Under certain model set-ups, the policy can appear to be appropriately priced. In addition, richer guarantees can be exhibited when using better market consistent set-up. We should also keep in mind that the replicating portfolio (hedging assets) should be calculated using the best market consistent setup. Sometimes, using your own economic 'good sense,' can help you make the best choice between different set-ups.

IV. NUMERIX ESG

Designed to reduce model risk by enabling precise calibration of multi-factor models, Numerix ESG allows the user to specify component models and correlation, and then apply a joint calibration process to accurately capture the observed market dynamics of each risk factor. Users can create unified hybrid models that capture correlation among economies and asset classes; and access scenarios throughout the enterprise across a wide range of interfaces.

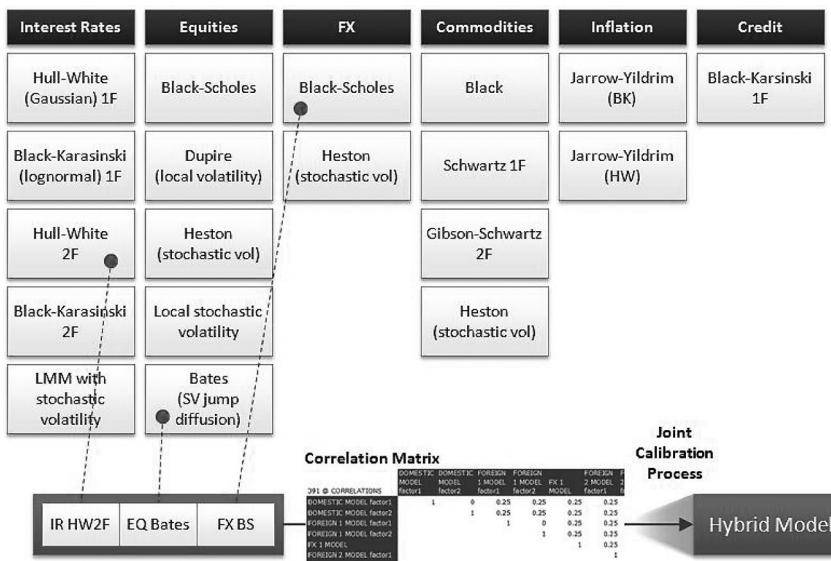
The sophisticated stochastic simulation framework can be used to produce risk neutral and real world economic scenarios that can be used across all business units, for all work products. Scenario sets can be easily integrated with in-house or other third party liability systems providing capital market model consistency across the firm.

With Numerix ESG, clients can find a competitive advantage through the application of sophisticated ESG techniques right from the product design phase. Full model transparency for different reporting frameworks including IFRS, GAAP, VACARVM, C3 Phase 2 and Solvency II is also possible.

V. THE NUMERIX HYBRID MODEL FRAMEWORK

Why the hybrid model framework? As insurance companies look at new ways to hedge risk in a way that is consistent with market-observed behavior, the key is bringing together interest rate, equity, volatility, credit and other factors within a unified, efficient model framework. To accomplish this, it is necessary to incorporate stochastic processes across multiple asset classes and factors. This requires a simultaneous calibration process to accurately capture correlation between volatility factors. Numerix has developed a unique hybrid model framework that provides a structure for designing and calibrating such a model.

The concept behind the hybrid model framework is that the best model is selected for each underlying. These “component” models are individually calibrated and then linked together through a correlation matrix that defines the “hybrid” model. A joint calibration is applied, allowing, volatility to be represented in a market-consistent manner. This approach offers a high degree of flexibility in the selection of various single- and multi-factor models across “n” economies for baskets of arbitrary size.



For more information about Numerix ESG, Numerix Hybrid Market Framework or Advanced Models and Indicators for ESG, please contact sales@numerix.com.

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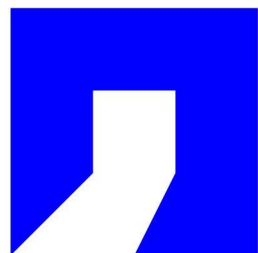
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Mr. Boukfaoui oversees development of insurance modeling solutions for the company. Prior to Numerix he was a quant and actuarial consultant both in the United States and Europe focusing on quantitative techniques for the modeling and assessment of Asset and Liability risk. Mr. Boukfaoui is a graduate from Ecole Polytechnique and holds a master of science in Financial Mathematics

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